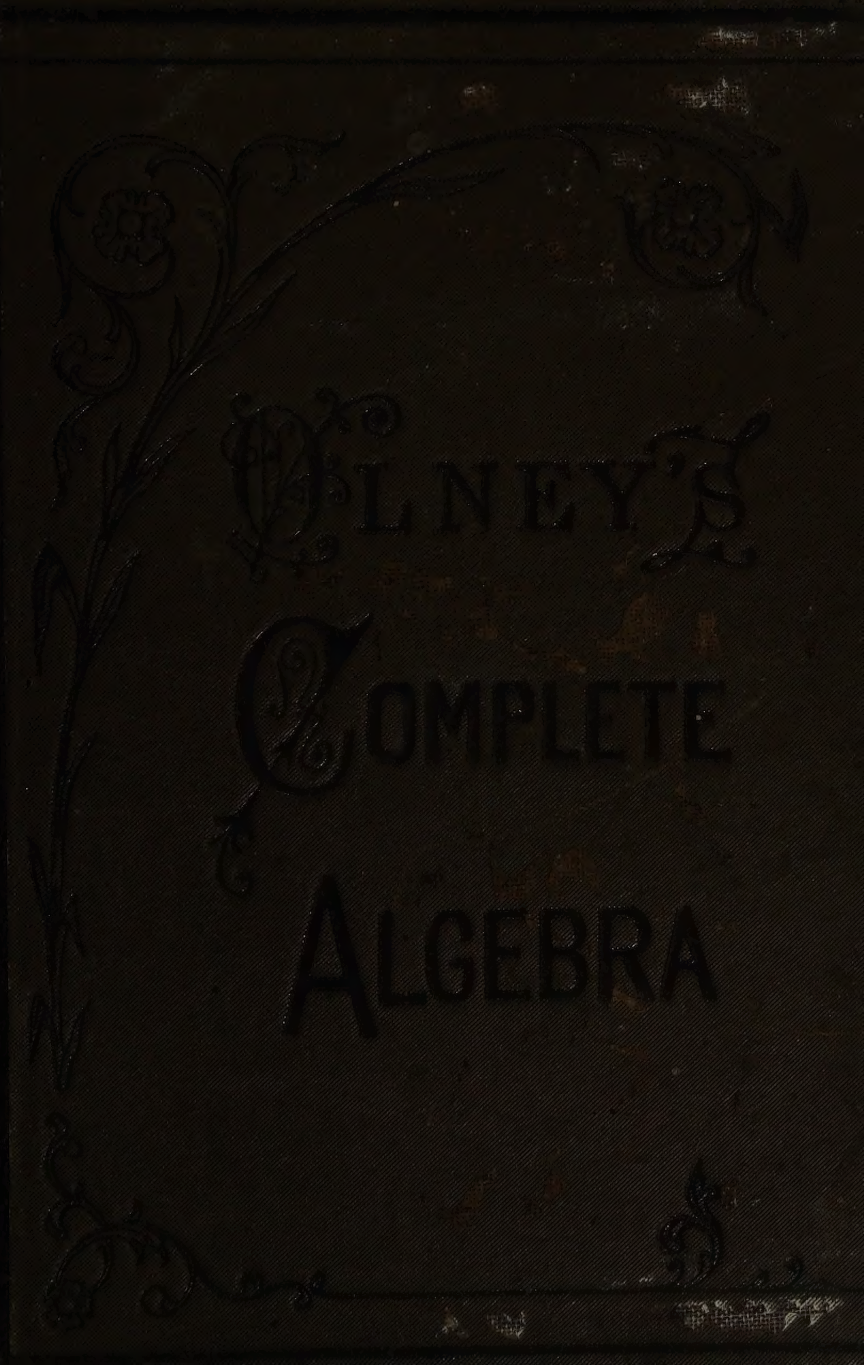

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PREFACE

THIS book is a new edition of the author's Complete Algebra. The changes made in the original work are not sufficient to preclude the use of the old edition in classes with the new. The matter of the new edition runs article for article, and, with very inconsiderable differences, page for page, with the old. The changes made consist in the breaking up of the longer definitions and rules into short paragraphs, for the convenience of teacher and pupil in the class-room, the simplification of a few of the more abstruse demonstrations, the omission of quite a large number of "suggestions" to teacher and to pupil, and the substitution of simpler problems for a few that teachers generally have thought too difficult. These changes, together with the larger and more open page, and the more beautiful typography, it is thought will render the book still more acceptable to teachers than it has hitherto been.

In effecting this revision the author has had the aid of several accomplished teachers who have been familiar with the practical workings of the book in the class-room. Of these, Prof. C. H. Churchill, of Oberlin College, has read the entire book and offered suggestions freely. With reference to the changes made in the problems, the author has been guided by the opinion of a large number of practical teachers who have used the book for years.

The same thoroughness in the discussion of principles, comprehensiveness, philosophic accuracy, and clearness of statement, and the same careful adaptation to training the pupil to think clearly and express his thoughts with propriety, characterize this edition as won for the first its great popularity. This book is, therefore, not a mere child's book, but is designed for pupils who have the knowledge of Arithmetic usually considered requisite for a commencement of this study. For less mature pupils the author has prepared another volume.

This treatise is designed to lay the foundation of a good mathematical education. Algebra develops the mathematical language, and is the great mathematical instrument. The Literal Arithmetic (see pages 1–210) is the basis of all mathematical training. Without a good measure of ability to handle the various forms of literal expressions—the more complex as well as the more simple—it is impossible to become a mathematician. Hence an unusual amount of care is bestowed upon this part of the work. A mere elementary knowledge of the simpler forms of numerical equations, is indeed easily obtained, but it is of very little worth. A thorough mastery of the elements of Algebra as presented in this volume is one of the most valuable acquisitions which can be secured in the schools, whether we consider its value as a mental discipline, or as a foundation for more advanced work.

The attempt to train the pupil to methods of reasoning, rather than in mere methods of operating, has given character to the presentation of every topic. Propositions are clearly stated at the outset, and demonstrations are given in form, and with the rigor of a geometrical argument. That there is some defect in our methods of instruction, in this regard, must be painfully evident to every one who has been called to examine large numbers of our youth in this study. The author has examined for admission to college, from 25 to 150 different students from all parts of our country, each year, for the last 27 years, and he has almost invariably found little or no knowledge of the processes *as arguments*, even when a good degree of skill in the use of the processes had been attained. Perhaps a majority of those examined could multiply the square root of 2 by the cube root of 3, but scarce one in 50 could develop the process in a logical form, or, in most cases, give any rational account of it. Now, it need not be said that, in a course of education, this is a fundamental defect; it is failure just where success is vital. The *processes* of a mathematical science are of comparatively little worth to a great majority of those who study them; the development of the reasoning powers to which such studies are addressed, is of the highest importance to all. By teachers who cannot appreciate these truths, this book will very probably be misunderstood; but to such as do feel the force of them, the author appeals with the fullest confidence, not indeed that his book will meet the exigency, but that it will be welcomed as an effort in the right direction, and as a help in remedying this radical defect.

The Introductory portion is found by many an adequate discipline for the younger and more immature pupils before entering upon the treatise proper, while the Appendix makes the volume sufficiently comprehensive for many of our colleges.

Grateful for the favor with which the original work, as well as all he has written, has been received; the author has been stimulated to perfect, as far as possible, the different members of the Series, with the hope of rendering them still more useful.

EDWARD OLNEY.

UNIVERSITY OF MICHIGAN,
Ann Arbor, *July*, 1881.

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INTRODUCTION TO ALGEBRA.

SECTION I.

HOW LETTERS ARE USED TO REPRESENT NUMBERS.

Ex. **1.** How many fives are 3 times 5 + 4 times 5 + 2 times 5? 9 Or 3 fives + 4 fives + 2 fives? Or $3 \times 5 + 4 \times 5 + 2 \times 5$? *Ans.*, 9 times 5, or 9 fives, or 9×5.

2. How many dozen are 2 dozen + 8 dozen + 5 dozen + 1 dozen?

3. How many score are 3 score, 5 score, 8 score, and 2 score?

4. Without performing any multiplications tell the sum of 2 times 8, 5 times 8, and 4 times 8.

5. Needles are put up in papers, with the same number in each paper; how many are 3 papers + 5 papers + 4 papers + 1 paper?

Suggestion.—In each of the above exercises a certain number of units is taken in a group, the number being the same in each group in any one example; thus in Ex. 1, the groups are 5's; in 2, a dozen; in 3, a score; in 4, they are 8's; in 5, some *certain number, but we do not need to know what number.*

NOTE.—This Introduction is designed for pupils who have not studied the author's "First Principles," and for whom the body of this work may be deemed too severe. Somewhat mature pupils, with a good knowledge of Arithmetic, will be able to commence at page 1 of the body of the work.

1. *In Algebra we often use letters to represent, or stand for, numbers.* The following exercises will show how:

6. Two times *a certain number* + 7 times the *same number* + 4 times the *same number*, are how many; *i. e.*, how many times that number?

7. Let a stand for a certain number. How many are 4 times $a + 3$ times $a + 2$ times a? *Ans.*, 9 times a.

8. If we let m represent the number of needles in 1 paper of needles, how many needles are there in 5 papers + 3 papers + 1 paper, or how many times m are $5m$'s, $3m$'s, and $1m$? If m represents 25, how many are $9m$?

2. *Thus we may use any letter to represent any number, only so that it always means the same number in the same exercise.*

3. *When a letter is used to represent a number, the figure which tells how many times the number represented by the letter is taken, is just written before the letter, the word "times" being left out.*

Illustration. $3a$ means 3 *times* a, $4b$ means 4 *times* b, $7m$ means 7 *times* m, $105x$ means 105 *times* the number represented by x, whatever that number may be.

4. *A* **Coefficient** *is a number placed before a letter to tell how many times the letter is taken.*

A coefficient is therefore a factor. Thus in $5a$, 5 and a are two factors of the number represented by $5a$. The letters are called *Literal* factors. The figures are called the *Numerical* factors. Thus in $5a$, 5 is the numerical factor. Letters may be used as coefficients as well as figures. If no figure stand before a letter, the letter is taken *once*, or its coefficient is said to be 1. Thus, m means one time m, and its coefficient is 1.

9. Mention the numerical coefficients and the literal factors in the following expressions:

(1.) $6a+5b+3m-4n$.
(3.) $a-b-12c$.
(2.) $3a-15m+22m-4b+x-y$.
(4.) $b+2c-m+4n-z$.

Queries.—Of what is 12 the coefficient in $12x$? Of what is x the literal factor? Of what is 12 the numerical factor?

10. What is the sum of $10a+5a+7a+2a$?* If a stand for 13 what is the sum?

Ans., $10a+5a+7a+2a = 24a = 24 \times 13 = 312$.

Query.—If the a in $10a$ meant one number, in the $5a$ *another* number, the a's in $7a$ and $2a$ still other numbers, could you answer this exercise in the same way? You could not answer it at all. The a must mean the same number all the time, in the same example.

11. $3a+2a+5a+8a$, are how many times a? *Ans.*, $18a$. How much is this if a is 6? *Ans.*, 108. How much if a is 11? *Ans.*, 198.

12. Eleven times 8, minus 5 times 8, are how many times 8?

13. $11a-5a$ are how many times a? Eleven times any number, minus 5 times the same number, are how many times that number?

14. $12x-7x =$ how many times x? How much is this if x represents 3? If x represents $2\frac{1}{5}$? *Ans.* to the last, 11.

15. $5b + 4b + 10b - 12b =$ how many times b? How much is $5b+4b+10b-12b$, if $b = 3\frac{2}{7}$?

16. How much is $3m+8m-4m+6m-5m-2m$? How much is this if $m = \frac{2}{3}$?

Solution. $3m+8m$ are $11m$, $11m-4m$ are $7m$, $7m+6m$ are $13m$, $13m-5m$ are $8m$, $8m-2m$ are $6m$. Hence the answer is $6m$. If $m = \frac{2}{3}$, $6m = 6 \times \frac{2}{3} = 4$.

---

* The pupil is presumed to be acquainted with the use of the signs. They are explained in the body of the full treatise, simply as a part of the science.

5. In such examples as the last you can add together all the quantities with the + sign into one sum, making in this case $17m$, and all those with the — sign into another, making in this case $11m$, and then subtract. Thus, $17m-11m$ is $6m$, the same result as before. This is, generally, the better way to solve such examples.

17. How much is $10x-4x-2x+3x-8x+11x$? How much is it if x is 3?

18. How much is $10x-15x$?

Suggestions.—Of course we cannot take $15x$ out of $10x$. But we can take $10x$ of the $15x$ from the first $10x$, and there will then remain $5x$ of the $15x$, which cannot be taken out of the $10x$. We indicate this by writing it $-5x$. This means that $5x$ was to be subtracted, but that we had nothing to take it from.

19. How much is $3x-5x-2x$? *Ans.*, $-4x$.

Query.—What does this answer mean?

20. How much is $12a+3a-5a-20a$? How much is this if $a=\frac{1}{2}$.

Query.—What does the answer mean?

21. How much is 2 times 3 times a certain number, as 5? *Ans.*, 6 times the number.

22. How much is 5 times $7m$? that is, 5 times 7 times a number which we will represent by m? *Ans.*, 35 times m, or $35m$.

23. How much is 6 times $8a$? 7 times $3a$? 10 times $7b$? 9 times $8y$? 9 times 8 times a number are how many times that number?

24. How much is 3 times $7m$, and 4 times $8m$, minus 2 times $22m$, if m represents 6? *Ans.*, $9m$, or 54.

25. What is $10a$ divided by 2; that is, what is $\frac{1}{2}$ of $10a$? $27x$ divided by 9 is how much?

Ans. to the last, $3x$.

26. How many times a number is 10 times that number, divided by 2; that is, $\frac{1}{2}$ of 10 times a number?

27. How much is $\frac{1}{8}$ of $48x$? $25x$ divided by 5? $\frac{1}{11}$ of $11x$? $11x$ divided by 11? $7x$ divided by 7?

28. Divide $10x$ by 5, then add $3x$, then multiply by 2, then subtract $4x$, then divide by 3? What is the result?

SECTION II.

COMBINATIONS OF LETTERS.

6. *When two letters representing numbers are written side by side, their product is indicated.*

Thus, ab means the product of the two numbers represented by a and b, or $a \times b$. $3abc$ means 3 times the product of the numbers represented by a, b, and c, or $3 \times a \times b \times c$.

[**Note.** Instead of saying, as above, *the number represented* by a, we usually simply say "the number a," or, "a," without using the word number at all. Thus we say 3 times the product of a, b, and c.]

7. *If we want to represent the product of a number represented by a letter, as* a, *by itself a certain number of times, instead of writing* aa, *or* aaa, *etc., as we might, we write* a^2, a^3, *etc.*

Thus b^4 means the same as $bbbb$. a^2 is read "a square;" a^3, "a cube;" b^4, "b fourth power;" x^5, "x fifth power;" etc.

[Do not read b^4 "b *fourth*," but "b fourth power." You will hereafter learn that b^{iv} is "b fourth," and is quite a different thing from b^4.]

The little figure placed at the right and a little above the letter is one form of what is called an **Exponent.**

Thus in m^2, 2 is the exponent of m, and m^2 means m times m. In $6x^3$, 3 is the exponent of x, and $6x^3$ means $6 \times x \times x \times x$.

[An exponent does not always indicate a product. It does this only when it is a whole number. Thus a^{-3} does not mean aaa; and in $a^{\frac{2}{3}}$, the exponent $\frac{2}{3}$ has quite a different meaning from the 3, or -3, above. These other meanings of exponents will be explained later in the work.]

Ex. 1. How much is $4a^2b$, if $a = 2$, and $b = 5$? How much is $3a^3b^2x$, if $a = 3$, $b = 2$, $x = 8$? How much is $a^5b^2y^4$, if $a = 2$, $b = 1$, $y = 3$?

2. How much is $a^2b^2 + 2ay - by$, if $a = 4$, $b = 3$, $y = 2$? How much is $3a^2by^3 - 2ay^2 + 5b$, the letters having the same values as before. How much is $5by - 2ab^3 + 4a^3y^2 - 2a$?

3. How many times ab^2 is $4ab^2 + 2ab^2 - 3ab^2$? How many times a^3y is $10a^3y + 4a^3y - 6a^3y - a^3y$?

4. How many times am^2y^3 is 4 times $3am^2y^3$? How much* is 6 times $2am^2y^3$? 4 times $7a^2b^2c^2$? 10 times $13mn^2x^3$?

5. How many times ax is $\frac{1}{5}$ of $20ax$? $\frac{1}{7}$ of $35ax$? $102ax$ divided by 3? How much is $\frac{1}{8}$ of $72a^2x^3$? $125x^2y^2$ divided by 25? $18ny^3$ divided by 9?

8. We have learned in arithmetic that representing numbers by the figures 1, 2, 3, 4, etc., is called **Arabic** or **Decimal Notation.** In like manner, representing numbers by the small letters of the alphabet, as $a, b, c, d, \text{- - - -} x, y$, etc., is called **Literal Notation.**

---

* This means the same as the preceding question.

The pupil will see that this *Literal Notation* is altogether a different thing from the *Roman Notation*, in which the seven capital letters, I, V, X, L, C, D, M, are used. Because the *Literal Notation* is so much used in Algebra, it is often called the *Algebraic Notation.* But this notation is just as much used in some other branches of Mathematics, as in Algebra.

9. A **Term** is a group of letters combined together by multiplication, or division, or both.

In the expression $5a^2x - 3a + \frac{x^2}{y} - \frac{5ax}{b}$, $5a^2x$, $3a$, $\frac{x^2}{y}$, and $\frac{5ax}{b}$, are *terms*.

A **Monomial** is an expression consisting of but *one* term.

A **Binomial** is an expression consisting of *two* terms connected by the sign $+$, or $-$.

A **Trinomial** has *three* terms.

A **Polynomial** is an expression with more than one term.

6. Point out the monomials, binomials, trinomials, and polynomials in the following: $2ax-3b^2$, $5xy-6cd+a-2y$, $3a^2m^2xy$, c^2-d^2, $a+m$, $a+b+c-d$, $225a^2b^2c^2d^2$, $abcd$, $a-b$, ab, $c-x^2y+ax$, x^2+y^2, $10a^2+3xy$, x^2-y^2, $3a^2mxy$, a^2-5x+3.

10. **Similar Terms** are terms which have the same literal factors affected with the same exponents.

7. Point out the similar terms among the following: $3ax^2$, $2ax$, $-5a^2x^2$, ax, $17ax^2$, $16cy^2$, $-12c^2y^2$, $3a^2x^2$, $-5cy^2$, $6cy^2$, $10ax^2$, c^2y^2, $3a^3x$, $3ax^3$, $51a^3x$.

11. **Positive Terms** are terms which have the sign $+$.

Negative Terms are terms which have the sign $-$.

When no sign is expressed, $+$ is understood.

SECTION III.

HOW NUMBERS ARE ADDED IN THE LITERAL NOTATION.

12. **Rule.**—*I. Write the expressions so that similar terms shall stand in the same column.*

II. Combine the terms in each column and write the result underneath with its own sign.

The polynomial thus found is the sum.

Ex. 1. Add $5ax-2cy$, $3ax+4cy$, $cy-2ax$, $-4ax-3cy$, $-ax+5cy$, and $2ax+2cy$.

Operation.—Having written the numbers so that similar terms fall in the same column, we may begin to add with any column we choose. Adding the right-hand column we find it makes $+7cy$, and write this sum underneath the column added. In like manner the other column makes $3ax$ (or $+3ax$), which, as it stands first, we write without any sign, as + will then be understood. The sum is $3ax+7cy$.

$$\begin{array}{rrrr}
 & 5ax & - & 2cy \\
 & 3ax & + & 4cy \\
- & 2ax & + & cy^{*} \\
- & 4ax & - & 3cy \\
- & ax & + & 5cy \\
 & 2ax & + & 2cy \\
\hline
 & 3ax & + & 7cy
\end{array}$$

(2.)

$$\begin{array}{rrrrr}
5cd & - & 2a & + & 4xy \\
2cd & + & 3a & - & 5xy \\
 & & 8a & - & 2xy \\
-6cd & & & + & 14xy \\
 & - & 3a & - & 7xy \\
 & + & 11a & + & xy \\
4cd & - & 15a & & \\
\hline
5cd & + & 2a & + & 5xy
\end{array}$$

(3.)

$$\begin{array}{rrrrr}
10am & - & 3dy^3 & + & 2a^2x \\
-6am & + & 4dy^3 & - & 10a^2x \\
4am & - & 8dy^3 & - & 2a^2x \\
7am & - & 13dy^3 & + & 6a^2x \\
-9am & + & 2dy^3 & - & 5a^2x \\
am & + & 18dy^3 & - & a^2x \\
\hline
7am & & & - & 10a^2x
\end{array}$$

---

* When no sign is expressed before a term, + is understood.

4. Add $5x-3a+b+7$ and $-4a-3x+2b-9$.
Sum, $2x-7a+3b-2$.

5. Add $2a+3b-4c-9$ and $5a-3b+2c-10$.

6. Add $3a+2b-5$, $a+5b-c$, and $6a-2c+3$.

7. Add $6xy-12x^2$, $-4x^2+3xy$, $4x^2-2xy$, and $-3xy+4x^2$.
Sum, $4xy-8x^2$.

8. Add $3a^2+4bc-e^2+10$, $-5a^2+6bc+2e^2-15$, and $-4a^2-9bc-10e^2+21$.

9. Add $5x^2y-3cx$, $4x^2y$, $15cx$, $13cy-ab$, $11ab-12cx-9x^2y$, and $-10ab-13cy$.

10. Add $7ax+3cy-5ax-cy+3cy-ax+11ax-5y+3c$.

11. Add $-3cx^3+5ay$, $ax-3y$, and $5-3x^2$.

SECTION IV.

HOW NUMBERS ARE SUBTRACTED IN THE LITERAL NOTATION.

13. Rule.—*Change the signs of the terms in the subtrahend, or conceive them to be changed, and add the result to the minuend.*

Ex. 1. From $5by-6a^2+3x^3$ subtract $2by+3a^2+x^3$.

Operation.

$5by-6a^2+3x^3$	*Minuend.*
$\underline{-2by-3a^2-x^3}$	*Subtrahend* with signs changed.
$3by-9a^2+2x^3$	The *Remainder* sought, which is the sum of the minuend and the subtrahend with its signs changed.

2. From $3ax+5b^2y^3-2m^2$ take $3ax-b^2y^3-3m^2$.
Rem., $6b^2y^3+m^2$.

3. From $10a-3b+2c-x^3$ subtract $b-5c+x^3$.
Rem., $10a-4b+7c-2x^3$.

4. From $12xy-3c^2+ab$ take $6xy+c^2-2ab$.

5. From $5c^2y-3ab$ take $mx-2c^2y$.
Rem., $7c^2y-3ab-mx$.

6. From $x^3-11xyz+3a$ take $-6xyz+7-2a-5xyz$.
Rem., x^3+5a-7.

14. When there is a term in the subtrahend which has no similar term in the minuend, we see that this term appears in the remainder with its sign changed.

7. From $6ac^2-2by^3+4x-3cy$ take $-2ac^2+3b^3y-3x-3cy+m$. *Rem.*, $8ac^2-2by^3-3b^3y+7x-m$.

8. From $8m^3x^3-3a-4b$ take a^2-b^2.
Rem., $8m^3x^3-3a-4b-a^2+b^2$.

9. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$.

10. From a^2+b^2 take a^2-b^2.

11. From $3cm-y$ take $2b-3c$.

12. From $x^2-2xy+1$ take $4xy$.

SECTION V.

HOW NUMBERS ARE MULTIPLIED IN THE LITERAL NOTATION.

15. **To multiply two Monomials together.**

Rule.—*I. Multiply the numerical coefficients. To this product affix the literal factors, affecting each letter with an exponent equal to the sum of the exponents of that letter in both terms.*

II. If the signs of the terms are alike, the sign of the product is +; if unlike, —.

Ex. 1. Multiply $5ax^2$ by $3a^2x^3$.

Operation.—The product of the numerical coefficients 3 and 5 is 15. To this we affix the letters a and x; and as a has an exponent 1[*] in the multiplicand, and 2 in the multiplier, we give it an exponent 3 in the product, and x an exponent 5 for a like reason. The product is therefore $15\,a^3x^5$.

$$\begin{array}{r} 5ax^2 \\ 3a^2x^3 \\ \hline 15a^3x^5 \end{array}$$

2. Multiply $10mn^2$ by $3m^2n^2$. $3xy$ by $4x^2y^2$. $7cx$ by $3cx$. $2a$ by a^3.

3. Multiply $-5a^2$ by $6b$. *Product*, $-30a^2b$.

4. Multiply $-3a^2x$ by $-2a^3x^2y$. *Product*, $6a^5x^3y$.

5. Find the products of the following: $-7a^2x$ by $2axy$; $10c^2mx^2$ by $-3m^2x$; $9a$ by $4b$; $-am$ by $-xy$; $3c^2d^2$ by $-ab$; $-5xy$ by $-x^2y^2$; $3ax^3$ by $-5a^2$; $-7xy$ by $-10x^3y$; $5abx$ by $-ca^2$; ax by cy.

16. To multiply two factors together when one or both are Polynomials.

Rule.—*Multiply all the terms of the multiplicand by each term of the multiplier, and add the products.*

Ex. 1. Multiply $2a^2x-3by+4m$ by $2a^3b^2m$.

Operation. — It is immaterial where we write the multiplier, but we may as well write it as in arithmetic. So also it makes no difference whether we begin at the

$$\begin{array}{r} 2a^2x-3by+4m \\ 2a^3b^2m \\ \hline 4a^5b^2mx-6a^3b^3my+8a^3b^2m^2 \end{array}$$

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[*] When no exponent is expressed, 1 is always understood.

right hand or the left to multiply. *It is customary to arrange the letters in a term alphabetically*; thus we write $4a^5b^2mx$ instead of $4b^2a^5xm$, or any such form. There would be *really* no difference in the *value of the terms*, however, in whatever order the letters came.

2. Multiply $6ax-5a^2x+3ax^3$ by $2a^2x^2$; $2my-3cy^2$ by $5m^2c^3$; $4ab-3cd+x$ by ay; $a+b-c$ by x; $10a^2m^2x-4any+3mx^3$ by $6a^2y^3$.

3. Multiply $3a^2x^2-2ay^3+5xy$ by a^2-2xy.

Operation.

$$\begin{array}{ll} & 3a^2x^2-2ay^3+5xy \\ & \qquad\qquad a^2-2xy \\ \cline{2-2} \text{Product by } a^2, & 3a^4x^2-2a^3y^3+5a^2xy \\ \text{Product by } -2xy, & \qquad\qquad\qquad\qquad\qquad -6a^2x^3y+4axy^4-10x^2y^2 \\ \cline{2-2} \text{Sum of partial prod's.,} & 3a^4x^2-2a^3y^3+5a^2xy-6a^2x^3y+4axy^4-10x^2y^2 \end{array}$$

There being no similar terms in these partial products, we can add them only by connecting them with their proper signs.

4. Multiply $x^2-2xy+y^2$ by $2x-3y$.

Operation.

$$\begin{array}{ll} & x^2-2xy+y^2 \\ & \qquad 2x-3y \\ \cline{2-2} \text{Product by } 2x, & 2x^3-4x^2y+2xy^2 \\ \text{Product by } -3y, & \qquad -3x^2y+6xy^2-3y^3 \\ \cline{2-2} \text{Entire product,} & 2x^3-7x^2y+8xy^2-3y^3 \end{array}$$

5. Multiply $a+n$ by $a-n$. *Prod.*, a^2-n^2.

6. Multiply $a^4+a^3+a^2+a+1$ by a^2-1.

7. Multiply $a^2-2ab+b^2$ by $a^2+2ab+b^2$.

8. Multiply $m+n$ by $m+n$.

9. Multiply $m-n$ by $m-n$.

10. Multiply $4x^2-9y^2$ by $x+2y$.

11. Multiply together $x-3$, $x-5$, $x-4$, and $x-7$.
Prod., $x^4-19x^3+131x^2-389x+420$.

12. Multiply $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.
Prod., $x^3+y^3+z^3-3xyz$.

SECTION VI.

HOW NUMBERS ARE DIVIDED IN THE LITERAL NOTATION.

17. **To divide Monomials when dividend and divisor consist of the same letter affected with exponents.**

Rule.—*Subtract the exponent of the divisor from that of the dividend. The common letter with this difference as an exponent is the quotient. If the signs of the divisor and dividend are alike, the sign of the quotient is + ; if unlike, −.*

Ex. **1.** Divide a^5 by a^3. *Quot.*, a^2.

The student will observe that the product of the divisor and quotient must always equal the dividend. In this case, $a^3 \times a^2 = a^5$.

2. Divide $-x^7$ by x^2. *Quot.*, $-x^5$.

3. Divide $-m^3$ by $-m^2$. *Quot.*, m.

4. Divide b^6 by $-b^4$. *Quot.*, $-b^2$.

5. Give the quotients in the following cases: $y^7 \div y^4$; $-x^{10} \div x^7$; $a^8 \div -a^5$; $-c^5 \div -c^3$.

18. **To divide one Monomial by another when there is no letter in the divisor which is not in the dividend, and no exponent in the divisor is greater than the exponent of the same letter in the dividend.**

Rule.—*Divide the numerical coefficients, and to the quotient annex the literal factors, affecting each with an exponent equal to its exponent in the dividend minus that in the divisor, and suppressing all factors whose exponents are 0. If the dividend and divisor have like signs, the quotient will have the sign + ; if unlike, −.*

Ex. 1. Divide $15b^3x^2y$ by $3bxy$. *Quot.*, $5b^2x$.

2. Divide $21a^2m^3ny^2$ by $-7a^2mny$. *Quot.*, $-3m^2y$.

3. Divide $-105a^3y^2$ by $-21a^2$. *Quot.*, $5ay^2$.

4. Divide $-18mn^2x$ by $6mnx$. *Quot.*, $-3n$.

Give the quotients in the following:

5. $12a^2y^3 \div 3ay^2$.

6. $-64a^5y^2 \div 16a^3y^2$.

7. $81x^4y^3 \div -27x^3y^3$.

8. $-35a^2b^2x^3 \div -7ab^2x^2$.

9. $24m^3y^2 \div -8m^3y^2$.

19. General rule for dividing one Monomial by another.

Rule.—*Write the divisor under the dividend in the form of a common fraction, and then reduce this fraction to its lowest terms by cancelling all factors common to both numerator and denominator.*

The sign of the quotient is determined as in the last case.

10. Divide $18a^2x^3$ by $4a^3x^2$.

Operation. $18a^2x^3 \div 4a^3x^2 = \frac{18a^2x^3}{4a^3x^2}$. Now in 18 and 4 there is a common factor 2 which can be cancelled; the a^2 in the numerator will cancel two factors of a from a^3 in the denominator, and leave a factor a in the denominator, and in like manner x^2 in the denominator cancels x^2 from the x^3 in the numerator, leaving x therein. Hence, $\frac{18a^2x^3}{4a^3x^2}=\frac{9x}{2a}$.

11. Divide $12m^2y^3$ by $10m^3y$.

Operation.

$$12m^2y^3\div 10m^3y=\frac{12m^2y^3}{10m^3y}=\frac{6y^2}{5m}.$$

12. Divide $48a^2x^2y^3$ by $-32a^2x^3y^2$. *Quot.*, $-\frac{3y}{2x}$.

13. Divide $-30b^5x^3y$ by $-40b^5x^4$. *Quot.*, $\frac{3y}{4x}$.

14. Divide $6ax$ by $4by$. *Quot.*, $\frac{3ax}{2by}$.

15. Divide $-11my$ by $10nx$. *Quot.*, $-\frac{11my}{10nx}$.

Ex's **16** to **21**. Find the quotients in the following cases: $24a^5x^2\div 15mx^2$; $-21ax^2$ by $-14am^2$; $-5ab$ by $5xy$; $16a^2y^2$ by $-12a^2y^2$; $7a^2x^3$ by $-3x^3$; $-4mn^3$ by $8mn$.

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20. To divide a Polynomial by a Monomial.

Rule.—*Divide each term of the dividend by the divisor; and write the quotients, connecting them with their own signs.*

22. Divide $6a^2x^4y^6-12a^3x^3y^6+15a^4x^5y^3$ by $3a^2x^2y^2$.

Operation.

$$\begin{array}{r|l} 3a^2x^2y^2 & 6a^2x^4y^6-12a^3x^3y^6+15a^4x^5y^3 \\ \hline & 2x^2y^4-4axy^4+5a^2x^3y \end{array}$$

23. Divide $12a^4y^6-16a^5y^5+20a^6y^4-28a^7y^3$ by $-4a^4y^3$.

24. Divide $15a^2bc-20acy^2+5cd^2$ by $-5abc$.

Quot., $-3a+\frac{4y^2}{b}-\frac{d^2}{ab}$.

25. Divide $16m^2x^3-24m^3x^5+10m^3x^3$ by $2m^2x^3$.

26. Divide $35a^2b^2y^2+28a^3b^2y^3-49ab^3y^4$ by $-7ab^2y^2$.

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21. To perform division when both dividend and divisor are Polynomials.

Rule.—*I. Arrange both dividend and divisor with reference to some letter found in both, i. e., place that term first at the left which has the highest exponent of this letter, the term containing the next highest exponent of this letter next, etc.*

II. Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

III. Multiply the divisor by this term of the quotient, and subtract the product from the dividend. To this remainder bring down as many terms of the dividend as may be necessary to form a new dividend.

IV. Divide the first term of this new dividend by the first term of the divisor, and proceed as before, repeating the process till the dividend is all used.

Ex. **1.** Divide $6a^2x^2-4ax^3-4a^3x+x^4+a^4$ by $-2ax+a^2+x^2$.

Operation.—Arranging dividend and divisor with reference to a, and proceeding according to the rule, we have the following:

$$\begin{array}{rllllll}
a^2-2ax+x^2\,) & a^4 & -4a^3x & +6a^2x^2 & -4ax^3 & +x^4 & (\,a^2-2ax+x^2 \\
 & a^4 & -2a^3x & +\ a^2x^2 & & & \\
\cline{2-4}
 & & -2a^3x & +5a^2x^2 & -4ax^3 & & \\
 & & -2a^3x & +4a^2x^2 & -2ax^3 & & \\
\cline{3-5}
 & & & a^2x^2 & -2ax^3 & +x^4 & \\
 & & & a^2x^2 & -2ax^3 & +x^4 & \\
\cline{4-6}
\end{array}$$

The polynomials might equally well have been arranged with reference to x. Thus $x^2-2ax+a^2\,)\,x^4-4ax^3+6a^2x^2-4a^3x+a^4$. This arrangement would give $x^2-2ax+a^2$ for the quotient, which is essentially the same as before.

2. Divide $x^2y^2+x^4+y^4$ by $xy+x^2+y^2$.
Quot., x^2-xy+y^2.

3. Divide $a^2+2ab+b^2$ by $a+b$.

4. Divide $a^2-2ab+b^2$ by $a-b$.

5. Divide $4ax+4x^2+a^2$ by $2x+a$.

6. Divide $a^2b^2+b^5+a^3b^2+a^5$ by a^2-ab+b^2.

7. Divide $10ac+3c^2+3a^2+4b^2+8ab+8bc$ by $2b+a+3c$.

8. Divide x^5-y^5 by $x-y$.

Operation.

$$
\begin{array}{l}
x-y\,)\,x^5-y^5\,(\,x^4+x^3y+x^2y^2+xy^3+y^4 \\
\qquad\quad \underline{x^5-x^4y\qquad} \\
\qquad\qquad\quad x^4y-y^5 \\
\qquad\qquad\quad \underline{x^4y-x^3y^2\qquad} \\
\qquad\qquad\qquad\qquad x^3y^2-y^5 \\
\qquad\qquad\qquad\qquad \underline{x^3y^2-x^2y^3\qquad} \\
\qquad\qquad\qquad\qquad\qquad\quad x^2y^3-y^5 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2y^3-xy^4\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad xy^4-y^5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{xy^4-y^5}
\end{array}
$$

9. Divide x^7-y^7 by $x+y$.

10. Divide $4a^4-b^4$ by $2a-b$.

11. Divide $20ax^4+4a^6-4x^6-25a^2x^2$ by $2a^3+2x^3-5ax$.

12. Show that $(a^3-b^3)\div(a^2+ab+b^2)=a-b$.

SECTION VII.

A LITTLE ABOUT FACTORING.

22. **The Factors** *of a number are those numbers which multiplied together produce it.*

Thus 3 and 4 are the factors of 12, because $3\times 4=12$.

So $a+b$ and $a-b$ are the factors of a^2-b^2, because $(a+b)\times(a-b)=a^2-b^2$. Try it, and see.

Ex. 1. What is the product of a and b? What are the factors of ab?

2. What is the product of 3, x, and y? What are the factors of $3xy$?

3. What does a^3 mean? What are the factors of a^3? What of a^3b^2? Of $5a^3b^2$?

Suggestion.—Such an expression as $5a^3b^2$ may be resolved in a great variety of ways: thus 5, a, a, a, b, and b are its factors; also, 5, a^3, and b^2; also, 5, a^2, a, and b^2; also, $5a$, a^2, b and b, etc.

4. What is the product of a and $x+y$? What are the factors of $ax+ay$?

5. What is the product of $3a$ and $a-b$? What are the factors of $3a^2-3ab$? Of $2a-2ab$?

THE SQUARE OF THE SUM OF TWO NUMBERS.

Ex. 1. What is the product of $a+b$ and $a+b$? What are the factors of $a^2+2ab+b^2$?

2. What is the product of $x+y$ and $x+y$? What are the factors of $x^2+2xy+y^2$?

3. What is the product of $1+x$ and $1+x$? What are the factors of $1+2x+x^2$?

23. We see from the last examples that *the square of the sum of two numbers equals the square of one of them, + twice the product of the two, + the square of the second.* Thus $(a+b)\times(a+b)$ is the square of the sum of the two numbers a and b, and is equal to $a^2+2ab+b^2$.

This principle, and those in 24 and 25, are of great importance in factoring.

THE SQUARE OF THE DIFFERENCE OF TWO NUMBERS.

Ex. 1. What is the product of $x-y$ and $x-y$? What are the factors of $x^2-2xy+y^2$?

2. What is the product of $m-n$ and $m-n$? What are the factors of $m^2-2mn+n^2$?

3. What is the product of $1-x$ and $1-x$? What are the factors of $1-2x+x^2$?

4. What is the product of $2-x$ and $2-x$? What are the factors of $4-4x+x^2$?

24. From these examples, we see that *the square of the difference of two numbers is equal to the square of one of them, — twice the product of the two, + the square of the other.* Thus $(x-y)\times(x-y)=x^2-2xy+y^2$.

THE PRODUCT OF THE SUM AND DIFFERENCE OF TWO NUMBERS.

Ex. 1. What is the product of $x+y$ and $x-y$? What are the factors of x^2-y^2?

2. What is the product of $a+b$ and $a-b$? What are the factors of a^2-b^2?

3. What is the product of $1+x$ and $1-x$? What are the factors of $1-x^2$?

4. What is the product of $2+x$ and $2-x$? What are the factors of $4-x^2$?

25. We see, from these examples, that *the product of the sum and difference of two numbers is equal to the difference of their squares.* Thus $(x+y)\times(x-y)=x^2-y^2$.

Ex. 1. What are the factors of $2a-2b$?

2. What are the factors of $3a^2-3a^3x$?

3. What are the factors of $c^2+2cd+d^2$?

4. What are the factors of $a^2-2am+m^2$?

5. What are the factors of a^2-c^2?

6. What are the factors of x^2-2x+1?

7. What are the factors of $9-x^2$?

8. What are the factors of a^2+2a+1?

SECTION VIII.

HOW OPERATIONS IN FRACTIONS ARE PERFORMED IN THE LITERAL NOTATION.

26. For the various operations in fractions in the *literal notation*, the ordinary rules of arithmetic for the corresponding cases apply, only, that the fundamental operations of addition, subtraction, multiplication, and division are performed by the preceding rules.

TO REDUCE FRACTIONS TO THEIR LOWEST TERMS.

27. *What is the rule for this operation in arithmetic?*

Ex. 1. Reduce $\frac{25a^2m^2x}{15am^3x^2}$ to its lowest terms. *Result*, $\frac{5a}{3mx}$.

2. Reduce $\frac{105b^2y^3}{15by^2}$ to its lowest terms.

3. Reduce $\frac{12a^2b^2-16a^3b}{8a^2bx+4a^5b^2}$ to its lowest terms.

Suggestion.—Divide numerator and denominator by $4a^2b$.

4. Reduce $\frac{28x^2y^2-42x^3y^2}{7x^3y}$ to its lowest terms.

5. Reduce $\frac{14a^2x^2-21ax^3+56a^3x^2}{28a^3x^3-35ax^4}$ to its lowest terms.

6. Reduce $\frac{a^2-x^2}{a^2+2ax+x^2}$ to its lowest terms.

Suggestion.—Try $a+x$ and see if it will not divide both terms of the fraction.

7. Reduce $\frac{a+b}{a^2-b^2}$ to its lowest terms. *Result,* $\frac{1}{a-b}$.

8. Reduce $\frac{m^2-n^2}{m^2-2mn+n^2}$ to its lowest terms.

Suggestion.—Will any monomial divide the terms of the fraction?

IMPROPER FRACTIONS REDUCED TO WHOLE OR MIXED NUMBERS.

28. What is the rule for this operation in arithmetic?

Ex. 1. Reduce $\frac{x^2-2ax+a^2}{x-a}$ to an integral form.

Result, $x-a$.

Suggestion.—Divide the numerator by the denominator.

2. Reduce $\frac{10a^2-13ax-3x^2}{2a-3x}$ to an integral form.

3. Reduce $\frac{12c^3+8ac^2x^2-3acx-2a^2x^3}{4c^2-ax}$ to an integral form.

4. Reduce $\frac{10a^2-13ax-3x^2}{5a+x}$ to an integral form.

5. Reduce $\frac{12c^3+8ac^2x^2-3acx-2a^2x^3}{3c+2ax^2}$ to an integral form.

6. Reduce $\frac{ay+b}{y}$ to an integral or mixed form.

Operation. $\begin{array}{l} y)\underline{ay+b}. \\ \quad a+\frac{b}{y} \end{array}$ The term ay can be divided by y, giving a. But we can only indicate the division of b by y, by writing it in the form $\frac{b}{y}$, and as the sign of both b and y is $+$, the quotient $\frac{b}{y}$ is $+$, and is to be added to a.

6. Reduce $\frac{20x^3-10x+4}{5x}$ to an integral or mixed form.

Result, $4x^2-2+\frac{4}{5x}$.

7. Reduce $\frac{2a^2x-x^3}{a}$ to an integral or mixed form.

Result, $2ax-\frac{x^3}{a}$.

Query.—Why has $\frac{x^3}{a}$ the $-$ sign?

8. Reduce $\frac{a^3+x^3-x^4}{a+x}$ to an integral or mixed form.

Result, $a^2-ax+x^2-\frac{x^4}{a+x}$.

9. Reduce $\frac{3ax+b^2}{x}$ to an integral or mixed form. Also, $\frac{ab-b^2}{x}$.

MIXED NUMBERS REDUCED TO IMPROPER FRACTIONS.

29. What is the rule for this operation in arithmetic?

Ex. 1. Reduce $a-\frac{b}{a}$ to the form of a fraction.

Operation.—The integral part is a. This multiplied by a makes a^2. From this b is to be *subtracted*, because of the $-$ sign. Thus we have for the result $\frac{a^2-b}{a}$.

2. Reduce $2x+\frac{3y}{x}$ to the form of a fraction. Also, $a-x-\frac{3a}{x}$.

3. Reduce $a+x-\frac{a^2+2ax+x^2}{a-x}$ to the form of a fraction.

Suggestions.—The integral part, $a+x$, multiplied by the denominator $a-x$, gives a^2-x^2. From this *subtracting* $a^2+2ax+x^2$, as the sign before the fraction indicates, we have $-2ax-2x^2$. Hence the result is $\frac{-2ax-2x^2}{a-x}$.

We may also write this thus:

$$a+x-\frac{a^2+2ax+x^2}{a-x}=\frac{a^2-x^2-(a^2+2ax+x^2)}{a-x}.$$

Now the quantity in the parenthesis is to be subtracted from the a^2-x^2. Hence, changing the signs of the terms in the parenthesis, and dropping the marks, we have $\frac{a^2-x^2-a^2-2ax-x^2}{a-x}$. Adding similar terms, this becomes $\frac{-2ax-2x^2}{a-x}$.

4. Reduce $1-\frac{a^2-2ab+b^2}{a^2+b^2}$ to the form of a fraction.

Result, $\frac{2ab}{a^2+b^2}$.

5. Reduce $a-x-\frac{a^2+x^2-5}{a+x}$ to the form of a fraction.

Result, $\frac{5-2x^2}{a+x}$.

6. Reduce $3x-\frac{4x^2-5}{5x}$ to the form of a fraction.

Result, $\frac{11x^2+5}{5x}$.

TO REDUCE FRACTIONS TO EQUIVALENT FRACTIONS HAVING A COMMON DENOMINATOR.

30. *Give the rules of arithmetic for this process.*

We shall use only the method of multiplying both terms of each fraction by the denominators of all the other fractions.

Ex. 1. Reduce $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}$ to equivalent fractions having a common denominator. *Results,* $\frac{ayz}{xyz}, \frac{bxz}{xyz}, \frac{cxy}{xyz}$.

2. Reduce $\frac{4}{by}, \frac{2b}{y}, \frac{3}{4}$ to equivalent fractions having a common denominator. Also, $\frac{a}{x+y}, \frac{b}{x-y}, \frac{c}{3}$.

Results of the last, $\frac{3ax-3ay}{3x^2-3y^2}, \frac{3bx+3by}{3x^2-3y^2}, \frac{cx^2-cy^2}{3x^2-3y^2}$.

TO ADD FRACTIONS.

31. *Repeat the rules of arithmetic for this purpose.*

Ex. 1. Add $\frac{x}{2}, \frac{x}{3}$, and $\frac{x}{5}$. *Sum,* $\frac{31x}{30}$.

2. Add $\frac{2x}{3}$, and $\frac{x+2}{2}$. *Sum,* $\frac{7x+6}{6}$.

3. Add $\frac{2-x}{5}$, and $\frac{3x-1}{2}$. *Sum,* $\frac{13x-1}{10}$.

4. Add $\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}$. *Sum,* $\frac{2a^2+2b^2}{a^2-b^2}$.

5. Add $\frac{1-x^2}{1+x^2}$, and $\frac{1+x^2}{1-x^2}$. *Sum,* $\frac{2+2x^4}{1-x^4}$.

TO SUBTRACT FRACTIONS.

***32.** What is the rule given in arithmetic for this purpose.*

Ex. 1. From $\frac{2x}{3}$ take $\frac{x}{4}$. *Rem.*, $\frac{5x}{12}$.

2. From $\frac{a}{3}$ take $\frac{b}{2}$. *Rem.*, $\frac{2a-3b}{6}$.

3. From $\frac{x-1}{3}$ take $\frac{x+2}{5}$. *Rem.*, $\frac{2x-11}{15}$.

4. From $\frac{1}{x-y}$ take $\frac{1}{x+y}$. *Rem.*, $\frac{2y}{x^2-y^2}$.

5. From $\frac{1-x^2}{1+x^2}$ take $\frac{1+x^2}{1-x^2}$. *Rem.*, $\frac{-4x^2}{1-x^4}$.

MULTIPLICATION OF FRACTIONS.

***33.** How is a fraction multiplied by a whole number?*

Ex. 1. Multiply $\frac{3a}{2b}$ by $5a^2$. *Prod.*, $\frac{15a^3}{2b}$.

2. Multiply $\frac{a-b}{3c+2}$ by $a+b$. *Prod.*, $\frac{a^2-b^2}{3c+2}$.

3. Multiply $\frac{1-x}{1-x^2}$ by $1-x$. *Prod.*, $\frac{1-x}{1+x}$.

4. Multiply $\frac{2c}{3a^2b}$ by $3a$. *Prod.*, $\frac{2c}{ab}$.

5. Multiply $3x$ by $\frac{a}{b}$. *Prod.*, $\frac{3ax}{b}$.

6. Multiply $\frac{a^2-2ab+b^2}{a^2+2ab+b^2}$ by $a+b$. *Prod.*, $\frac{a^2-2ab+b^2}{a+b}$.

7. Multiply $10a^2$ by $\frac{15x}{6a^3}$. *Prod.*, $\frac{25x}{a}$.

8. Multiply $\frac{x}{2}$ by 6; by 8; by 10.

9. Multiply $\frac{x-1}{3}$ by 3; $\frac{2x+3}{5}$ by 5.

***34.** Give the rule for multiplying one fraction by another.*

Ex. 1. Multiply $\frac{a}{3b}$ by $\frac{5}{2b^2}$. *Prod.*, $\frac{5a}{6b^3}$.

2. Multiply $\frac{a}{b}$ by $\frac{x}{y}$. *Prod.*, $\frac{ax}{by}$.

3. Multiply $\frac{c-d}{5a}$ by $\frac{c+d}{2-a}$. *Prod.*, $\frac{c^2-d^2}{10a-5a^2}$.

4. Multiply $\frac{2x+3}{5}$ by $\frac{10x}{7}$. *Prod.*, $\frac{4x^2+6x}{7}$.

5. Multiply $\frac{a^2-b^2}{3ab}$ by $\frac{2a^2}{a-b}$. *Prod.*, $\frac{2a^2+2ab}{3b}$.

DIVISION OF FRACTIONS.

***35.** How is a fraction divided by a whole number?*

Ex. 1. Divide $\frac{3a}{5x}$ by $2x^2$. *Quot.*, $\frac{3a}{10x^3}$.

2. Divide $\frac{6a^2}{13b^3}$ by $2a^2b$. *Quot.*, $\frac{3}{13b^4}$.

3. Divide $\frac{a^2-b^2}{x+y}$ by $a-b$. *Quot.*, $\frac{a+b}{x+y}$.

4. Divide $\frac{a^2-b^2}{x+y}$ by $x-y$. *Quot.*, $\frac{a^2-b^2}{x^2-y^2}$.

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36. *How is any quantity divided by a fraction?*
Ans. *By multiplying it by the divisor inverted.*

Ex. 1. Divide 6 by $\frac{2a}{3b}$. *Quot.*, $\frac{9b}{a}$.

2. Divide $\frac{a}{5}$ by $\frac{3}{x}$. *Quot.*, $\frac{ax}{15}$.

3. Divide $\frac{2-c}{3}$ by $\frac{4}{2+c}$. *Quot.*, $\frac{4-c^2}{12}$.

4. Divide $\frac{a-1}{c+d}$ by $\frac{m-n}{x+y}$.
Quot., $\frac{ax+ay-x-y}{cm+dm-cn-dn}$.

5. Divide $\frac{x^2-2xy+y^2}{ab}$ by $\frac{x-y}{bc}$. *Quot.*, $\frac{cx-cy}{a}$.

6. Divide $\frac{2x^2}{a+b}$ by $\frac{3x^3}{a^2-b^2}$. *Quot.*, $\frac{2a-2b}{3x}$.

SECTION IX.

HOW PROBLEMS ARE SOLVED IN ALGEBRA.

Ex. 1. John is 3 times as old as James, and the sum of their ages is 32; how old is each?

Solution.—To solve this by Arithmetic, we reason thus: Since John's age is 3 times James's, both their ages together make 4 times James's age, and this is 32 years. Now 4 times James's age = 32 years. Hence, James's age is $\frac{1}{4}$ of 32, or 8 years; and John's age being 3 times James's, is 3×8, or 24 years.

To solve it by Algebra, we proceed as follows: Let x represent James's age; then, since John is 3 times as old, $3x$ will represent his; the sum of their ages will be $3x+x$. Now since the sum of their ages is 32, $3x+x=32$; or, $4x=32$. If $4x=32$, $x=\frac{1}{4}$ of 32, or $x=8$. Hence, James's age is 8, and John's being $3x$, is 3×8, or 24.

37. The expression $3x+x=32$ *is what is called an* **Equation**; *and it is by the use of equations that we solve problems in Algebra.*

2. A merchant said that he had 72 yards of a certain kind of cloth, in three rolls. In the first roll, there were a certain number of yards; in the second, 3 times as many as in the first; and in the third, twice as many as in the first. How many yards were there in the first roll?

Suggestions.

The equation is $x+3x+2x=72$

Now, $x+3x+2x$ is $6x$, hence $6x=72$

And if $6x=72$, x, or $1x$, is $\frac{1}{6}$ of 72, $x=12$

Queries.—What does the x stand for? *Answer.* The number of yards in the first roll. In this problem which is the most, $x+3x+2x$, or 72? To start with, do you know how much x is? Then is it a *known*, or an unknown quantity, at the outset?

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38. The answer to a problem is represented in the beginning of the solution by one of the latter letters of the alphabet, usually x, if there is need of but one letter, and is called the **Unknown Quantity**.

3. A boy on being asked how old he was, replied, "If you add to my age 3 times my age, and 5 times my age, and subtract twice my age, the result will be 49 years. How old was he?

Suggestions.

The equation is $x+3x+5x-2x=49$

Hence, since $x+3x+5x-2x$ is $7x$, $7x=49$

If $7x=49$, $x=\frac{1}{7}$ of 49, or $x=7$

4. There are three times as many girls as boys in a party of 60 children. How many boys are there? How many girls?

5. In a barrel of sugar weighing 200 lbs., there are three varieties, A, B, and C, mixed. There is 7 times as much of B as of A, and twice as much of C as of A. How much of A is there? How much of each of the other kinds?

Ans., Of A, 20 lbs.; of B, 140 lbs.; of C, 40 lbs.

6. There were 4 kinds of liquor put into a cask, 2 times as much of the second as of the first, 2 times as much of the third as of the second, and 2 times as much of the fourth as of the third. The cask sprang a leak, and three times as much leaked out as was put in of the first kind, when it was found that there were 36 gallons remaining. How much was there put in of each kind?

Suggestion.—The equation is $x+2x+4x+8x-3x=36$.

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7. In a pasture there are a certain number of cows and 23 sheep, in all 34 animals. How many cows are there?

Solution.—As it is the number of cows we seek, let x represent the number of cows. Then $x+23$ is the number of animals in the pasture, and the equation is

$$x+23=34.$$

Now the $x+23$ means just the same thing as the 34, that is $x+23=34$. So, if we subtract 23 from each, there will be just as much left of one as of the other. Subtracting 23 from $x+23$, there remains x, and subtracting 23 from 34, there remains 11. Hence $x=11$; that is, there are 11 cows.

39. The part of an equation on the left of the sign = is called the **First Member**, and that on the right, the **Second Member.**

Note.—The pupil must not think because these examples are so simple that he can "do them in his head," that therefore algebra is a very clumsy method of solving problems. He will find by and by, that though the equation does not really help any in the solution of such simple questions, it will solve a great many very difficult ones about which he might puzzle his brains a great while to no purpose, if algebra did not come to his aid. Stick to it, then, and learn how to use this new instrument, the *Equation*, and you will by and by find it wonderfully useful. It is a grand patent for solving problems.

8. In a certain pasture there are three times as many horses as cows, and 20 sheep. In all there are 100 animals. How many cows are there? How many horses?

Suggestions.

The equation is $x+3x+20=100$

Subtracting 20 from each member, $x+3x=80$

Uniting the terms of the first member, $4x=80$

Dividing each member by 4, $x=20$

Hence there are 20 cows; and, as there are three times as many horses as cows, there are 60 horses.

9. In a basket of 60 apples there are 4 times as many red apples as yellow, and 10 green apples. How many yellow apples are there? How many red?

10. John and James together have 75 cents. James has 25 cents less than John. How many cents has John?

Suggestions.—Let x represent the number of cents which John has. Then, as James has 25 cents less, $x-25$ will represent what he has. But both together have 75 cents. Hence the equation is

$$x+x-25=75.$$

Now, if we drop the -25 from the first member, we make this member 25 greater than it now is, *i. e.*, $x+x$ is 25 greater than

$x+x-25$. Therefore, if we add 25 to the second member, making it 100, the members will still be equal.*

This gives $\quad x+x=100$
or, $\quad 2x=100$
Hence $\quad x=50$

That is, John has 50¢.

11. A merchant has 90 yards of cloth in two pieces. The longer piece lacks ten yards of containing 3 times as much as the shorter. How much in each piece?

Suggestion.—The equation is $x+3x-10=90$.

12. Divide the number 50 into two parts so that one part shall lack 10 of being 5 times the other.

Suggestions.—The parts are represented by x, and $5x-10$. They are 10 and 40.

13. Divide the number 50 into 3 parts, such that the second shall be 5 more, and the third 15 less than the first.

The parts are 20, 25, and 5.

Suggestion.—The equation is $x+x+5+x-15=50$.

14. There are 52 animals in a field. Twice the number of cows + 11 is the number of sheep, and 3 times the number of cows — 13 is the number of horses. How many of each kind? *Ans.*, 9 cows, 29 sheep, and 14 horses.

15. A man said of his age,

"If to my age there added be
One-half,† one-third, and three times three,
Six score and ten the sum will be.
What is my age? Pray show it me."

Suggestion.—The equation is $x+\frac{x}{2}+\frac{x}{3}+9=130$.

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* The word "Transposition" is purposely omitted from this introduction. Nor is the idea designed to be presented. It will be better for the pupil to "think out" the process, as above.

† Meaning "one-half *my age*," "one-third *my age*."

Subtracting 9 from each member,

$$x+\frac{x}{2}+\frac{x}{3}=121.$$

Now, we can get rid of the fractions in the first member by multiplying it by 6, the product of both the denominators. Thus, 6 times the first member is $6x+3x+2x$. Then, if we also multiply the second member by 6, the products will be equal. For if two quantities are equal, 6 times one of them is equal to 6 times the other.

Hence we have $6x+3x+2x=726$

Uniting terms, $11x=726$

Dividing by 11, $x=66$

16. Mary gave half her books to Jane, and one-third of them to Helen, when she had but 2 left. How many had she at first?

Suggestions.—Let x represent the number of books Mary had at first. Then she gave Jane $\frac{x}{2}$, and Helen $\frac{x}{3}$ books. And what she gave the other girls, added to what she had left, makes all she had in the first place. Hence the equation is

$$\frac{x}{2}+\frac{x}{3}+2=x.$$

Multiplying each member by 6, $3x+2x+12=6x$

Subtracting $5x$ from each member, $12=x$

That is, Mary had 12 books at first.

17. A boy lost 25 cents of some money which his uncle gave him, and gave half he had left to his brother. He then earned 50 cents, when he had just as much as his uncle gave him. How much did his uncle give him?

Suggestions.—Let $x=$ the number of cents his uncle gave him. Then he had $x-25$ cents after losing 25 cents. After giving away half of this, he had the other half, or $\frac{x-25}{2}$ cents, left. He then earned 50 cents, and the amount he had was equal to what his uncle gave him.

Hence the equation is $\frac{x-25}{2}+50=x$

Multiplying each member by 2, $x-25+100=2x$

Uniting, $-25+100$ makes 75, and $x+75=2x$

Subtracting x from each member, $75=x$

18. A boy being asked how many marbles he had, said, "If I had five more than I have, half the number subtracted from 30 would leave twice as many as I now have." How many marbles had he?

Suggestions.—Letting x represent the number of marbles the boy had, the equation is

$$30-\frac{x+5}{2}=2x.$$

Now there is a little peculiarity about this equation, which the pupil must be careful to notice whenever it occurs, or he will make a great many mistakes. It is this: When we multiply each member by 2, to get rid of the fraction, we must write $60-x-5=4x$. The mistake would be to write $60-x+5=4x$. The explanation is, that the $-$ sign before the fraction $\frac{x+5}{2}$ does not belong to the x, but to the fraction *as a whole*. The sign of x in the fraction $-\frac{x+5}{2}$ is $+$, since when no sign is expressed $+$ is understood. What then becomes of the $-$ sign before the fraction, if it is not the same as the sign of x in the equation $60-x-5=4x$? It has been dropped, since the thing signified by it has been performed, and the $-$ sign before the x is the sign of that term in the original equation, changed. In like manner we subtract $+5$, by changing its sign.

The boy had 11 marbles.

19. What is the value of x in the equation $3x-\frac{2-2x}{3}=21\frac{1}{3}$?

Suggestions.—Multiplying each member by 3,

We have $9x-2+2x=64$

Hence, $x=6$

20. Find the value of x in the equation

$x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$. *Ans.*, $x=5$.

21. What is the value of x in the equation

$\frac{x-1}{3}-\frac{x+4}{5}=15-\frac{x+3}{4}$? *Ans.*, $x=40\frac{3}{23}$.

22. Show that in

$$\frac{x-1}{7}=7-\frac{23-x}{5}-\frac{4+x}{4},\quad x=8.$$

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23. Two boys were to divide 32 marbles between them so that $\frac{1}{2}$ of what one had should be 5 less than what the other had. How many was each to have?

Suggestions.—Letting $x=$ what one had,
then $32-x=$ what the other had.

The equation is $\frac{x}{2}+5=32-x$,

or $\frac{32-x}{2}+5=x$.

Query.—Why will either equation answer the purpose?

24. What number is that to which if 7 be added, half the sum will be 8 more than $\frac{1}{5}$ of the remainder of the number after 3 has been subtracted?

Equation $\frac{x+7}{2}-\frac{x-3}{5}=8$. *Ans.*, $x=13$.

25. The sum of two numbers is sixteen, and the less number divided by three is equal to the greater divided by five. What are the numbers?

Suggestion.—Let x and $16-x$ represent the numbers.

26. Divide twenty-two dollars between A and B, so that if one dollar be taken from three-fourths of B's share, and three dollars be added to one-half of A's money, the sums shall be equal. How many dollars will each have?

27. The sum of two numbers is thirty-three. If one-sixth of the greater be subtracted from two-thirds of the less number, the remainder will be seven. What are the numbers?

28. The sum of A's and B's money is thirty-six dollars. If five-eighths of B's, less two dollars, be taken from three-fourths of A's, the difference will be seven dollars. How many dollars has each?

29. The difference between two numbers is twenty-five; and if twice the less be taken from three times the greater, the remainder will be eighty. What are the numbers?

30. A and B gain money in trade, but A receives ten dollars less than B. If A's share be subtracted from twice B's, the remainder will be fifty-seven dollars. How much money did each receive?

31. One number is four less than another, and if twice the less be subtracted from five times the greater, the remainder will be thirty-eight. What are the numbers?

32. Two farms belong to A and B. A has twenty acres less than B. If twice A's number of acres be taken from three times B's, the remainder will be one hundred. How many acres has each?

33. One number is seven less than another, and if three times the less be taken from four times the greater, the remainder will be six times the difference between the two numbers. What are the numbers?

34. Anna is four years younger than Mary. If twice Anna's age be taken from five times Mary's, the remainder will be thirty-five years. What is the age of each?

35. One number is ten less than another. If three times the less be taken from five times the greater, the remainder will be seven times the difference of the two numbers. What are the numbers?

INTRODUCTION.

SECTION I.

A BRIEF SURVEY OF THE OBJECT OF PURE MATHEMATICS AND OF ITS SEVERAL BRANCHES.

1. **Pure Mathematics** is a general term applied to several branches of science, which have for their object the investigation of the properties and relations of quantity—comprehending number, and magnitude as the result of extension—and of form.

2. **The Several Branches** of Pure Mathematics are Arithmetic, Algebra, Calculus, and Geometry.

3. Arithmetic, Algebra, and Calculus treat of number, and Geometry treats of magnitude as the result of extension.

4. **Quantity** is the amount or extent of that which may be measured; it comprehends number and magnitude.

The term quantity is also conventionally applied to symbols used to represent quantity. Thus 25, m, XI, etc., are called quantities, although, strictly speaking, they are only representatives of quantities.

It is not easy to give a philosophical account of the idea or ideas, represented by the word *Quantity* as used in Mathematics; and doubtless, different persons use the word in somewhat different senses. It is obviously incorrect to say that "Quantity is anything which can be measured." Quantity may be affirmed of any such concept; nevertheless, it is not the thing itself, but rather the *amount* or *extent* of it. Thus, a load of wood, or a piece of ground, can be measured; but no one would think of the wood or piece of ground as being the quantity. The *quantity* (of wood or ground) is rather the *amount* or *extent* of it.

The word is very convenient as a general term for mathematical concepts, when we wish to speak of them without indicating whether it is number or magnitude that is meant. Thus we say, "m represents a certain quantity," and do not care to be more specific.

As applied to number, perhaps the term conveys the idea of the whole, rather than of that whole as made up of parts. It is, therefore, scarcely proper to speak of multiplying by a *quantity;* we should say, by a number. On the other hand, when we apply the term quantity to magnitude, it is with the idea that magnitude may be measured, and thus expressed in number.

The distinction between quantity and number is marked by the questions, "How much?" and "How many?"

5. **Number** is quantity conceived as made up of parts, and answers to the question, "How many?"

Thus, a distance is a quantity; but, if we call that distance 5, we convert the notion into number, by indicating that the distance under consideration is made up of parts. Again, m may mean a value, as of a farm. We may or may not conceive it as a number (as of dollars). If we think of it simply in the aggregate, as the worth of a farm, m represents quantity; if we think of it as made up of parts (as of dollars) it is a number.

6. Number is of two kinds. **Discontinuous** and **Continuous.**

7. **Discontinuous Number** is number conceived as made up of finite parts; or it is number which passes from one state of value to another by the successive additions or subtractions of finite units; *i. e.*, units of appreciable magnitude.

8. **Continuous Number** is number which is conceived as composed of infinitesimal parts; or it is number which passes from one state of value to another by passing through all intermediate values, or states.

Number, as considered in Arithmetic, and in this volume, is **Discontinuous Number.** Thus 5 grows till it becomes 9, by taking on additions of units of some conceivable value; as when we consider it as passing from 5 to 9, thus, 5, $5+1$ or 6, $6+1$ or 7, $7+1$ or 8, $8+1$ or 9. If the increment were any fraction, however small, *the form of the conception would be the same.*

Time affords a good illustration of *Continuous Number.* We usually conceive time as a *discontinuous number*, as when we think of it as made up of hours, days, weeks, etc. But it is easy to see that such is not the way in which time actually grows. A period of one day does not grow to be a period of one week by taking on a whole day at a time, or a whole hour, or even a whole second. It grows by imperceptible increments (additions). These inconceivably small parts, by which time is actually made up, we call infinitesimals; and number, when conceived as made up of such infinitesimals, we call **Continuous Number.**

9. **Arithmetic** treats of **Discontinuous Number,**—of its nature and properties, of the various methods of combining and resolving it, and of its application to practical affairs.

The leading topics of Arithmetic are:

1. **Notation**; *i. e.*, methods of representing number, as by the characters, 1, 2, 3, 4, etc., or by letters, as, a, b, m, n, x, y, etc.

2. **Properties of Numbers** or deductions from the methods of Notation.

3. **Reduction,** as from one scale to another, from one denomination to another, from one fractional form to another, or, in short, from any one form of expression to another equivalent form.

4. The various **methods of combining number,** as by addition, multiplication, and involution.

5. **Resolving Number,** as by subtraction, division, and evolution.

And all these processes as affected by the use of any notation, and upon integral or fractional discontinuous numbers of any kind.

Arithmetic, therefore, philosophically considered, embraces much that is usually classed as Algebra. Thus all that usually precedes Simple Equations, and all that is embraced in this Part I, is simply a repetition and extension of the processes of Arithmetic with a new notation—the literal. Again, logarithms are nothing but a new scheme of notation, by means of which certain combinations are more readily effected; and the making of logarithms is but a reduction from one form of expression to an equivalent one in another notation. In the ordinary notation, a certain number is represented thus, 256; in the logarithmic notation it is 2.40824.

10. **Algebra** treats of the *Equation*, and is chiefly occupied in explaining its nature and the methods of transforming and reducing it, and in exhibiting the manner of using it as an instrument for mathematical investigation.

The whole province of the relations of quantity, continuous or discontinuous number, is covered by Algebra, so far as the equation can be made the instrument of investigation. Much, therefore, of what is found in our Arithmetics can be more expeditiously treated by Algebra. Such are the subjects of Ratio, Proportion, the Progressions, Percentage, Alligation, etc. *In fact, the equation is the grand instrument of mathematical investigation, and demonstrates its efficiency in every department of the science.* To hope to get on in mathematics without Algebra, is to expect to walk without feet.

11. **Calculus** treats of *Continuous Number*, and is chiefly occupied in deducing the relations of the infinitesimal elements of such number from given relations between finite values, and the converse process, and also in pointing out the nature of such infinitesimals and the methods of using them in mathematical investigation.

12. **Geometry** treats of *magnitude* and *form* as the result of extension and position.

The principal divisions of the science of Geometry are:

1. The Ancient, Special, or direct Geometry (the common Geometry of our schools), including Trigonometry, Conic Sections, and all other geometrical inquiries conducted upon these methods.

2. The Modern, Indirect, or General Geometry (usually called Analytical), and

3. Descriptive Geometry.

SYNOPSIS.

Subject of Section.
Definition of Pure Mathematics: Several Branches of.
Subject Matter of the Several Branches.
Quantity: Uses of the term.
Number: Illustration of: Kinds of: Illustration of kinds.
Arithmetic: Topics of.
Algebra—Calculus—Geometry.

SECTION II.

LOGICO-MATHEMATICAL TERMS.

***13.* A Proposition** is a statement of something to be considered or done.

Illustration.—Thus, the common statement, "Life is short," is a proposition; so, also, we make, or state a proposition, when we say, "Let us seek earnestly after truth."—"The product of the divisor and quotient, plus the remainder, equals the dividend," and the requirement, "To reduce a fraction to its lowest terms," are examples of Arithmetical propositions.

14. Propositions are distinguished as *Axioms*, *Theorems*, *Lemmas*, *Corollaries*, *Postulates*, and *Problems*.

***15.* An Axiom** is a proposition which states a principle that is so simple, elementary, and evident, as to require no proof.

Illustration.—Thus, "A part of a thing is less than the whole of it," "Equimultiples of equals are equal," are examples of axioms.

If any one does not admit the truth of axioms, when he understands the terms used, we say that his mind is not sound, and that we cannot reason with him.

16. **A Theorem** is a proposition which states a real or supposed fact, whose truth or falsity we are to determine by reasoning.

Illustration.—"If the same quantity be added to both numerator and denominator of a proper fraction, the value of the fraction will be increased," is a *theorem.*

17. **A Demonstration** is the course of reasoning by means of which the truth or falsity of a theorem is made to appear. The term is also applied to a logical statement of the reasons for the processes of a rule. A solution tells *how* a thing is done; a demonstration tells *why* it is so done. A demonstration is often called *proof.*

18. **A Lemma** is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

Illustration.—Thus, in order to demonstrate the rule for finding the greatest common divisor of two or more numbers, it may be best first to prove that "A divisor of two numbers is a divisor of their sum, and also of their difference." This theorem, when proved for such a purpose, is called a *Lemma.*

The term *Lemma* is not much used, and is not very important, since most theorems, once proved, become in turn auxiliary to the proof of others, and hence might be called lemmas.

19. **A Corollary** is a subordinate theorem which is suggested, or the truth of which is made evident, in the course of the demonstration of a more general theorem, or which is a direct inference from a proposition.

Illustration.—Thus, by the discussion of the ordinary process of performing subtraction in Arithmetic, the following *Corollary* might be suggested: "Subtraction may also be performed by addition, as we can readily observe what number must be added to the subtrahend to produce the minuend."

20. **A Postulate** is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done. We may or may not know how to perform the operation.

Illustration.—Quantities of the same kind can be added together.

21. **A Problem** is a proposition to do some specified thing, and is stated with reference to developing the method of doing it.

Illustration.—A problem is often stated as an incomplete sentence, as, "To reduce fractions to forms having a common denominator."

22. **A Rule** is a formal statement of the method of solving a general problem, and is designed for practical application in solving special examples of the same class. Of course a rule requires a demonstration.

23. **A Solution** is the process of performing a problem or an example. It should usually be accompanied by a demonstration of the process.

24. **A Scholium** is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

Illustration.—Thus, after having discussed the subject of multiplication and division in Arithmetic, the remark that "Division is the converse of multiplication," is a scholium.

SYNOPSIS.

Subject of Section.
Proposition. *Ill.*—Varieties of.
Axiom. *Ill.*—One who will not admit the truth of.
Theorem. *Ill.*
Demonstration. Difference between a solution and a demonstration.
Lemma. *Ill.*—Why the term is unimportant.
Corollary. *Ill.*
Postulate. *Ill*
Problem. How stated. *Ill.*
Rule.
Solution.
Scholium. *Ill.*

PART I.

LITERAL ARITHMETIC

CHAPTER I.

FUNDAMENTAL RULES.

SECTION I.

Notation

25. **A System of Notation** is a system of symbols by means of which quantities, the relations between them, and the operations to be performed upon them, can be more concisely represented than by the use of words.

SYMBOLS OF QUANTITY.

26. In Arithmetic, as usually studied, numbers are rep-

NOTE.—Part First treats of the familiar operations of Addition, Subtraction, Multiplication, Division, Involution and Evolution, and the theory of Fractions. The only difference between the processes here developed and those with which the pupil is already familiar, grows out of the notation. Hence appears the appositeness of the term Literal Arithmetic. Hence, also, the teacher should be careful that the pupil see the unity of purpose, and the reason for any difference in method of execution.

resented by the characters, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, called Arabic figures, or, simply, figures.

27. In other departments of mathematics than Arithmetic, numbers or quantities are more frequently represented by the common letters of the alphabet, a, b, c, . . . m, n . . . x, y, z. These letters may, however, be used in Arithmetic; and the Arabic figures are used in all departments of mathematics.

This method of representing quantities by letters is often called the *Algebraic* method, and the method by the Arabic characters, the Arithmetical. It would be better to call the former the *Literal* method, and the latter the *Decimal.*

28. **The Literal Notation** has some very great advantages over the decimal for purposes of mathematical reasoning:

1st. The symbols are more general in their signification; and

2d. We are enabled to detect the same quantity anywhere in the process, and even in the result. Thus it happens that the processes become general formulas, or rules, instead of special solutions.

Illustration.—To illustrate the first statement, suppose we say a boy has 7 apples; you know *just how many* are meant. But when we say a boy has b apples, nobody can tell how many he has. In fact, it is not designed to tell the exact number, but only to say that he has *some* number. Again, 7 represents the same number of units always; but a letter may be used to represent any number of units we please; or, it may be used, as we have just said, without our caring to specify any precise number of units. This may seem to be a very unsatisfactory kind of notation; but with patience its advantages will appear. The following examples will illustrate the general, or comprehensive character of the literal notation more fully.

EXAMPLES.

Ex. 1. A boy has 8 marbles which he sells for three cents each, and takes his pay in pencils at 6 cents each. How many pencils does he receive?

Suppose we answer that he receives 3 times 8 divided by 6, or $\frac{3 \times 8}{6}$ pencils, without giving the number more explicitly.

Now take a similar example, using the literal notation; thus, A boy sells a *certain* number of marbles which we will represent by c, for a number of cents each, which we will call m, and takes his pay in pencils at b cents each. How many pencils does he receive?

We will answer as before, and say he receives c times m, divided by b, or $\frac{c \times m}{b}$ pencils.

The pupil will notice this difference between the answers; both, as they now stand, simply tell what operations to perform in order to get the answers; but, in the former case, we can perform the operations and get the explicit answer, 4, while in the latter case, we can only leave it as it is.

Such an answer as $\frac{c \times m}{b}$ may seem to the pupil to be no answer at all; and indeed it is not an answer in the same sense as he has been accustomed to think of answers; nevertheless it is often more useful. Notice that the answer 4 is only true for the specific example, while the answer $\frac{c \times m}{b}$ *is true in every like example.*

We also observe that the quantities 3, 8, and 6 do not appear distinctly in the numerical answer, 4; but the c, m, and b do in the literal, and would, in general, however complicated the problem. The literal answer is equivalent to the rule, *Multiply the price of one marble by the number of marbles, and divide the product by the price of a pencil.*

2. One boy sold 5 pears at 3 cents each; another sold 6 apples at 2 cents each; and a third sold 3 melons at 8 cents each. How much did they all receive?

Ans., 51 cents.

3. One boy sold b pears at c cents each; another sold m apples at n cents each; a third sold d melons at g cents each. How much did they all receive?

Ans., $b\times c+m\times n+d\times g$ cents.

Suggestions.—Notice that in the 3d example the several quantities of the problem are distinctly seen in the answer, but not so in the answer to *Ex.* 2. Moreover, the answer to *Ex.* 3 is equally true for any and all values of b, c, m, n, d, and g. Consider in like manner the two following:

4. If I buy 5 cords of wood at 4 dollars per cord, and pay for it in cloth at 2 dollars per yard, how many yards are required?

Ans., 10 yards.

5. If I buy a cords of wood at b dollars per cord, and pay for it in cloth at c dollars per yard, how many yards are required?

Ans., $\frac{a\times b}{c}$ yards.

6. A man had a flock of m sheep. He lost $2n$ of them and raised $10a$. After which he sold the flock at $\$c$ per head, taking his payment in cloth at $\$b$ per yard. What operations must be performed on these numbers in order to ascertain the number of yards of cloth received? And how will this number be represented?

Ans., $\frac{c\,(m-2n+10a)}{b}$.

7. A man bought 3 horses. He gave for the first twice as much as for the second, and for the third c times as much as for both the others. If x represents the price of the first, how much did he give for the third?

Ans., $\left(x+\frac{x}{2}\right)c$.

29. In using the decimal notation certain *laws* are established in accordance with which all numbers can be represented by the ten figures. Thus, it is agreed that when several figures stand together without any other mark, as 435, the right hand figure shall signify units, the second to the left, tens, the third, hundreds, etc.; also that the *sum* of the several values shall be taken. This number is, therefore, 4 hundreds + 3 tens + 5 (units).

30. In like manner, certain laws are observed in representing numbers by letters.

FIRST LAW.

Known Quantities, that is such as are given in a problem, are represented by letters taken from the first part of the alphabet; while **Unknown Quantities**, or quantities whose values are to be found, are represented by letters taken from the latter part of the alphabet.

Illustration.—A grocer has two kinds of tea, one of which is worth a cents (any *given* number being meant by a) per pound, and the other b cents. *How many pounds of each* must he take to make a chest of c pounds, which will be worth d dollars? In this problem, a, b, c, and d are the given or *known* quantities, and hence are represented by letters from the first part of the alphabet. The *unknown* or *required* quantities are the *number of pounds* of each of the *two* kinds of tea. We therefore represent the number of pounds of the first kind by x, and of the second kind by y.

Scholium.—This law is not very rigidly adhered to, except that letters after and including v, are *generally* used to represent *unknown* quantities, while the others are used for *known* quantities. But it is sometimes convenient to use a different notation. Thus, in problems in Interest, the principal may be represented by p, whether it is known or unknown, the interest, in like manner, by i, the rate per cent. by r, the time by t, etc.

Accented letters, as a', a'', a''', a'''', etc., (read "a prime," "a second," "a third," etc.), and letters with subscripts, as a_1, a_2, a_3, a_4, etc., (read "a sub 1," "a sub two," etc.), are sometimes used. This form of notation is used when there are several *like* quantities in the same problem, but which have different numerical values. Thus, in a problem in which several walls of different heights, breadths, and lengths, are considered, we may represent the several heights by a', a'', a''', etc., or a_1, a_2, a_3, etc.; the thicknesses by b', b'', b''', etc., or b_1, b_2, b_3, etc., and the lengths by l', l'', l''', etc., or l_1, l_2, l_3, etc.

The Greek letters are also often used both for known and unknown quantities.

The student will notice a difference between Algebra and Arithmetic, in that, in Algebra, the unknown quantities (what he has called the "Answers" in Arithmetic) are represented in solving a problem, and are used in the solution just like known quantities. This device gives Algebra a great advantage over Arithmetic.

SECOND LAW.

When letters are written in connection, without any sign between them, their product is signified. Thus abc signifies that the three numbers represented by a, b, and c are to be multiplied together.

Scholium I.—There is here an interesting difference between this notation and the decimal. There are two points of difference; 1st, The place of a letter, as at the right or left, has nothing to do with its value; and 2d, The sign understood between them is that of *multiplication* instead of *addition*, as in the decimal notation. In the decimal system, if we write the three figures 5, 4, and 3, as we have written the letters a, b, and c, thus 543; 1st, Each figure has a particular value dependent upon the place it occupies, the 5 representing *hundreds*, the 4 *tens*, and the 3 *units;* 2d, The amount represented is $500+40+3$. Moreover, the several letters are not at all restricted in signification; a may represent 5, 48, 10.06, or any other number, however small or however large, and integral, fractional, or mixed; and the same is true of any other letter. In fact, the meaning usually is, that a represents *any* number, b any *other*, and c any other, etc. The same letter, however, means the same thing throughout one problem. Such expressions as abc, $mnxy$, etc., are read by simply naming the letters in order.

Scholium 2.—When figures are written in connection with letters, their relation to the letters is the same as that of the letters to each other. Thus $4ab$ means the continued product of 4, a, and b. Also $125xy$ means the continued product of 125, x, and y.

31. A character like a figure 8 placed horizontally, ∞, is used to represent what is called *Infinity*, or a quantity larger than any assignable quantity.

Scholium.—By an infinite quantity is not meant one larger than any other, or the largest possible quantity. It simply means a quantity larger than any assignable quantity; *i. e.*, larger than any one which has limits. Thus, a series of 1s, as 1 1 1, etc., repeated without stopping, represents an infinite quantity, because, from the method of conceiving the quantity, it is necessarily greater than any quantity which we can assign or mention. If we assign a row of 9s reaching around the world, though it is an inconceivably great number, it is not as great as a series of 1s extending *without limit.* Moreover, one infinite may be larger than another; for a series of 2s extending without limit, as 2 2 2 2, etc., is *twice as large* as a series of 1s conceived in the same way. It is never of any use to try to comprehend the magnitude of an infinite quantity; we can not do it; although we can *compare* infinites just as well as finites.

SYMBOLS OF OPERATION.

32. **The Symbols of Operation** used in Algebra are the same as in Arithmetic, or in any other branch of mathematics; but, to refresh the memory, we will repeat them.

33. The perpendicular cross, $+$, is called the *plus* sign, and read "plus." It signifies that the quantities between which it is placed are added together. Thus, $a+2cb+xm$ is read, "a plus $2cb$ plus xm."

34. A short horizontal line, —, is called the *minus* sign, and is read "minus." It signifies that the quantity before which it is placed is to be subtracted. Thus, $a-2cb-xm+12ax$ is read, "a minus $2cb$ minus xm plus 12 ax."

35. An S-shaped symbol placed horizontally, $\sim$, is sometimes used to signify the difference between two quantities. $a\sim b$ is read, "the difference between a and b." This sign differs from the preceding in that it does not indicate which of the two quantities is to be taken as the subtrahend, while the *minus* sign requires us to consider the quantity before which it is placed as the subtrahend.

36. The oblique cross, ×, and a simple dot, ·, are each signs of *multiplication*. In the case of literal factors, the sign is usually omitted, according to the second law of notation. Thus, $4\times a\times c$, $4\cdot a\cdot c$, and $4ac$, signify exactly the same thing.

37. The signs of *division* are, a horizontal line between two dots, having the dividend at the left and the divisor at the right, as $12ac\div 2b$; or the dots without the line, as $12ac:2b$; or the line without the dots, the dividend being written above and the divisor below it, as $\frac{12ac}{2b}$; each of which is read, "$12ac$ divided by $2b$." In performing division, the divisor is sometimes written at the left of the dividend and separated from it by a curved line; the quotient is then written at the right and separated from the dividend in the same manner: as, $2a)12ac(6c$, in which $2a$ is the divisor, $12ac$ the dividend, and $6c$ the quotient. Sometimes, especially in Algebra, the divisor is written on the right of the dividend, and separated from it by a vertical line, the quotient in this case being written under the divisor. Thus, $\begin{array}{r|l} 12ac & 2a \\ \cline{2-2} & 6c \end{array}$ is the last example above, expressed in a different form.

Scholium.—It is very inelegant, though quite common in some parts of the country, to read such expressions as $\frac{12ac}{2b}$, "$12ac$ *over* $2b$." We should read "$12ac$ *divided* by $2b$."

[**Note.**—Let great care be taken that the nature of *exponents*, as explained in the succeeding articles, be clearly comprehended. No little difficulty arises from an imperfect understanding of this notation. A very common, though very erroneous method of reading such expressions, greatly aggravates the difficulty.]

***38.* A Power** of a number is the product which arises from multiplying the number by itself, *i. e.*, taking it a certain number of times as a factor.

***39.* A Root** of a number is one of several *equal* factors into which the number is to be resolved.

***40.* An Exponent** is a small figure, letter or other symbol of number written at the right and a little above another figure, letter or symbol of number.

***41.* A Positive Integral Exponent** signifies that the number affected by it is to be taken as a factor as many times as there are units in the exponent. It is a kind of symbol of multiplication.

Illustration. 2^3 (read, "2, third power"), signifies that two is to be taken three times as a factor, *i. e.*, $2 \times 2 \times 2$, or 8. 3^4 (read, "3, fourth power"), signifies $3 \times 3 \times 3 \times 3$, or 81; 81 is the fourth power of 3, because it is the product of 3 taken four times as a factor. a^5 is $aaaaa$. x^m, read "x, mth power," or "x, exponent m," is xxx . . . etc., till m factors of x are taken.

***42.* A Positive Fractional Exponent** indicates a power of a root, or a root of a power. The denominator specifies the root, and the numerator the power of the number to which the exponent is attached.

Illustration. $8^{\frac{2}{3}}$ (read “8, exponent $\frac{2}{3}$,” not “8, $\frac{2}{3}$ power,” there is no such thing as a $\frac{2}{3}$ power), is the second power of the third root of 8. The third root of 8 being 2, and the second power of 2 being 4, $8^{\frac{2}{3}} = 4$. We may also understand the power to be taken first, and then the root, as will be demonstrated hereafter. Thus, $8^{\frac{2}{3}}$ is the third root of the second power of 8. The second power of 8 is 64, the third root of which is 4, which is the same result as was obtained by taking the root first, and then the power. $(125)^{\frac{1}{3}}$ is 5. $(125)^{\frac{2}{3}}$ is 25. $(32)^{\frac{3}{5}}$ is 8. $x^{\frac{m}{n}}$ (read “x, exponent m divided by n”) means that x is to be resolved into n equal factors, and the product of m such factors taken.

43. A Negative Exponent, either integral or fractional, signifies the reciprocal of what the expression would be if the exponent were positive.

Illustration. 3^{-4} (read “3, exponent -4”) signifies $\frac{1}{3^4}$, or $\frac{1}{81}$. 2^{-3} is $\frac{1}{2^3}$, or $\frac{1}{8}$. x^{-a} is $\frac{1}{x^a}$, etc. Also $8^{-\frac{2}{3}}$ is $\frac{1}{8^{\frac{2}{3}}}$, or $\frac{1}{4}$. $m^{-\frac{a}{b}}$ is $\frac{1}{m^{\frac{a}{b}}}$.

44. The Radical Sign, $\sqrt{\ }$, is also used to indicate the square root of a quantity. When any other than the square root is to be designated by this, a small figure specifying the root is placed in the sign. Thus $\sqrt[3]{5}$ signifies the 3d, or cube root of 5, and is the same as $5^{\frac{1}{3}}$. $\sqrt[5]{34ab^3}$ indicates the 5th root of $34ab^3$ and is the same as $(34ab^3)^{\frac{1}{5}}$.

Scholium.—Read $\sqrt{5ac}$ (“the square root of $5ac$,” not “radical $5ac$”). The latter expression is generic, and applies as well to $\sqrt[3]{5ac}$, or $\sqrt[5]{5ac}$. Besides it is inelegant.

Let the pupil read the following examples and give the signification of each.

EXAMPLES.

Ex. 1. $25a^{-2}b^{\frac{2}{3}}$. Read, "25, a exponent -2, b exponent $\frac{2}{3}$." It means 25 multiplied by $\frac{1}{a^2}$, multiplied by the square of the cube root of b. (See articles **109, 110.**)

2. $x^{\frac{m}{n}-1}$. Read, "x, exponent $\frac{m}{n}-1$." Since $\frac{m}{n}-1$ is $\frac{m-n}{n}$, $x^{\frac{m}{n}-1}$ is the same as $x^{\frac{m-n}{n}}$, and hence means that x is to be raised to a power indicated by $m-n$, and the nth root of this power extracted.

3. Read and explain $2a^{\frac{1}{5}}b^{-6}$. $12x^m y^{-\frac{n}{c}}$.

SYMBOLS OF RELATION.

45. The Sign of Geometrical Ratio is two dots in the form of a colon, : . Thus $a : b$, is read "a is to b," or, "the ratio of a to b." It means the same as $a \div b$.

46. The Sign of Arithmetical Ratio is two dots placed horizontally, ·· . Thus $a \cdot\cdot\, b$ is read, "the Arithmetical ratio of a to b and is equivalent to $a-b$.

47. The Sign of Equality is two parallel horizontal lines, $=$. Thus, $2cx = xy$, is read, "$2cx$ equals xy." $5ac - 2by = 3x^2$ is read, "$5ac$ minus $2by$ equals $3x^2$."

Four dots in the form of a double colon, ::, is the sign of equality between ratios. Thus, $a : b :: c : d$, read, "a is to b as c is to d," means that the ratio of a to b equals the ratio of c to d, and may just as well be written $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, all of which expressions mean exactly the same thing.

48. The Sign of Inequality is a character somewhat like a capital V placed on its side, $<$, the opening being towards the greater quantity. Thus $a > b$ is read, "a greater than b." $m < n$ is read, "m is less than n."

49. The Sign of Variation is somewhat like a figure 8 open at one end and placed horizontally. Thus, $a \propto \frac{c}{d}$ is read, "a varies as $\frac{c}{d}$."

SYMBOLS OF AGGREGATION.

50. A Vinculum is a horizontal line placed over several terms, and indicates that they are to be taken together. The parenthesis, ( ), the brackets, [ ], and the brace, { }, have the same signification.

Illustration. $\overline{a+b}\times\overline{cd-e}$ means that $(a+b)$ is to be multiplied by $(cd-e)$. $(a+b)\times(cd-e)$ also means the product of $(a+b)$ and $(cd-e)$. Brackets and braces are used when one parenthesis would fall within another. Thus, $\{z+[a+(b+c)\,x]\,y\}\,u$, signifies that the product of $(b+c)$ multiplied by x, is to be added to a, and this sum multiplied by y; to this product z is to be added and the sum multiplied by u.

51. A vertical line after a column of quantities, each having its own sign, signifies that the aggregate of the column is to be taken as one quantity. Thus $\left.\begin{array}{r}+a\\-b\\+c\end{array}\right|\,x$ is the same as $(a-b+c)\,x$.

SYMBOLS OF CONTINUATION.

52. A series of dots, ..., or of short dashes, - - - - - -, written after a series of expressions, signifies "&c." Thus $a : ar : ar^2 : ar^3 \;.\;.\;.\;.\;.\;.\; ar^n$ means that the series is to be extended from ar^3 to ar^n, whatever may be the value of n

SYMBOLS OF DEDUCTION.

53. Three dots, two being placed horizontally and the third above and between, $\therefore$, signify *therefore*, or some analogous expression. If the third dot is below the first two, $\because$, the symbol is read "since," "because," or by some equivalent expression.

POSITIVE AND NEGATIVE QUANTITIES.

54. Positive and Negative are terms primarily applied to concrete quantities which are, by the conditions of a problem, opposed in character.

Illustration.—In estimating the value of a person's estate, his *property* may be called positive, and his *debts* negative. Distance *up* may be called positive, and distance *down*, negative. Time *before* a given period may be called positive, and *after*, negative. Degrees *above* 0 on the thermometer scale are called positive, and *below*, negative.

55. The signs + and — are used to indicate the *character* of quantities as positive or negative, as well as for the purpose of indicating addition and subtraction. (See article 57.)

56. In problems in which the distinction of positive and negative is made, each quantity is to be considered as having a *sign of character*, expressed or understood, besides the plus or minus sign, which indicates whether it is to be added or subtracted. The positive sign need not be written to indicate character, as it is customary to consider quantities whose character is not specified as positive.

Illustration 1.—In the expression $ab+m-cx$, let the problem out of which it arose be such, that a, m, and x, tend to a positive result, and b and c to an opposite, or a negative result. Giving these quantities their signs of character, we have, $(+a) \times (-b) + (+m) - (-c) \times (+x,)$ which may be read, "positive a multiplied by negative b, plus positive m, minus negative c multiplied by positive x." Suppressing the positive sign, this may be written, $a(-b) + m - (-c)x$, by also omitting the unnecessary sign of multiplication.

Illustration 2.—As this subject is one of fundamental importance, let careful attention be given to some further illustrations. We are to distinguish between discussions of the relations between mere abstract quantities, and problems in which the quantities have some concrete signification. Thus, if it is desired to ascertain the sum or difference of 468, or m, and 327, or n, as mere numbers, the question is one concerning the relation of abstract numbers, or quantities. No other idea is attached to the expressions than that each represents a certain number of units. But, if we ask how far a man is from his starting point, who has gone, first, 468, or m, miles directly east, and then 327, or n, miles directly west; or if we ask what is the difference in time between 468, or m, years B. C., and 327, or n, years A. D., the numbers 468, or m, and 327, or n, take on, besides their primary signification as quantities, the additional thought of *opposition in direction.* They therefore become, in this sense, concrete.

Again, a company of 5 boys are trying to move a wagon. Three of the boys can pull 75, 85, and 100 pounds each; and they exert their strength to move the wagon east. The other two boys can pull 90 and 110 pounds each; and they exert their strength to move the wagon west. It is evident that the 75, 85, and 100 are quantities having an opposite tendency from 90 and 110. Again, suppose a party rowing a boat up a river. Their united strength would propel the boat 8 miles per hour if there were no current; but the force of the current is sufficient to carry the boat 2 miles per hour. Which way will the boat move, and how fast? The 8 and 2 are quantities of opposite character in their relation to the problem. Once more, in examining into a man's business, it is found that he has a farm worth m dollars, personal property worth n dollars, and accounts due him worth c dollars. There is a mortgage on his farm of b dollars, and he owes on account a dollars. The m, n, and c are quantities opposite in their nature to b and a. *This opposition in character is indicated by calling those quantities which contribute to one result positive, and those which contribute to the opposite result negative.*

57. Purely abstract quantities have, properly, no distinction as positive and negative; but, since in such problems the plus or additive, and the minus or subtractive *terms* stand in the same relation to each other as positive and negative quantities, it is customary to call them such.

Illustration.—In the expression, $5ac-3cd+8xy-2ad$, though the quantities, a, c, d, x and y be merely abstract, and have no proper signs of character of their own, the *terms* do stand in the same relation to each other and to the result, as do positive and negative quantities. Thus, $5ac$ and $8xy$ tend, as we may say, *to increase* the result; while $-3cd$, and $-2ad$ tend *to diminish* it. Therefore the former may be called positive *terms*, and the latter negative.

58. **Scholium.**—*Less than zero.* Negative quantities are frequently spoken of as "less than zero." Though this language is not philosophically correct, it is in such common use, and the thing signified is so sharply defined and easily comprehended, that it may possibly be allowed as a conventionalism. To illustrate its meaning, suppose in speaking of a man's pecuniary affairs it is said that he is worth "less than nothing;" it is simply meant that his debts exceed his assets. If this excess were \$1000, it might be called negative \$1000, or –\$1000. So, again, if a man were attempting to row a boat up a stream, but with all his effort the current bore him down, his progress might be said to be less than nothing, or negative. In short, in any case where quantities are reckoned both ways from zero, if we call those reckoned one way greater than zero, or positive, we may call those reckoned the other way "less than zero," or negative.

59. The value of a Negative Quantity is conceived to *increase* as its numerical value *decreases.*

Illustration.—Thus $-3 > -5$, as a man who is in debt \$3, is better off than one who is in debt \$5, other things being equal. If a man is striving to row up stream, and at first is borne down 5 miles an hour, but by practice comes to row so well as only to be borne down 3 miles an hour, he is evidently gaining; *i. e.*, -3 is an *increase* upon -5. Finally, consider the thermometer scale. If the mercury stands at 20° below 0, (marked $-20°$) at one hour, and at $-10°$ the next hour, the temperature is increasing; and, if it increase sufficiently will become 0, *passing which* it will reach $+1°$,

+2°, etc. *In this illustration, the quantity passes from negative to positive by passing through* 0. This is assumed as a fundamental truth of the doctrine of positive and negative quantities, viz.: THAT A QUANTITY IN PASSING THROUGH 0 MAY CHANGE ITS SIGN.

[It appears in geometry, that a quantity may also change its sign in passing through *infinity*. Thus the tangent of an arc less than 90° is positive; but if the arc continually increases, the tangent becomes infinity at 90°, passing which it becomes negative.]

NAMES OF DIFFERENT FORMS OF EXPRESSION.

60. **A Polynomial** is an expression composed of two or more parts connected by the signs plus and minus, each of which parts is called a *term*.

61. **A Monomial** is an expression consisting of one term.

A Binomial is a polynomial having two terms.

A Trinomial is a polynomial having three terms.

Illustration. $5a^2b-cd^{\frac{1}{2}}+x-4(a+b)$ is a polynomial of 4 terms. The first three are monomial terms, and the last is a binomial term. $5ac-ef$ and x^2+y^2 are examples of binomials. $2a^2x^{\frac{1}{2}}y-125d^{-m}+12$ is a trinomial.

62. **A Coefficient** of a term is that factor which is considered as denoting the number of times the remainder of the term is taken.

The numerical factor, or the product of the known factors in a term is most commonly called the coefficient, though any factor, or the product of any number of factors in a term may be considered as coefficient to the other part of the term.

Illustration.—In the term $6a$, 6 is the coefficient of a. In ax, a may be called the coefficient of x, or 1 may be called the coefficient of ax. In $6axy$, 6 is the coefficient of axy, $6a$ of xy, and $6ax$ of y. In $5(ab-c)$, 5 is the coefficient of $(ab-c)$; and in $(2a^2-cd)xy$, $(2a^2-cd)$ is the coefficient of xy.

63. Similar Terms are such as consist of the same letters affected with the same exponents.

Illustration. $5ab$, $13ab$, and ab are similar. $-16x^2y^{\frac{1}{2}}$, $5x^2y^{\frac{1}{2}}$, and $-x^2y^{\frac{1}{2}}$ are also similar to each other. $4ab$, $5ab^3$, and $-2ab^2$ are dissimilar, as are $8ax$, $-5bx^{\frac{1}{2}}$, $4cx^3$, and $5xy$.

EXERCISES IN NOTATION.

Ex. **1.** Write in mathematical symbols, 5 times the square root of a, added to the cube of the sum of a and b.

Result, $(a+b)^3+5\sqrt{a}$, or $(a+b)^3+5a^{\frac{1}{2}}$.

How many terms in this result? What kind of a term is the first one?

2. Write the second power of a, plus 3 times the product of c square multiplied by b, diminished by m times the cube root of the binomial, the square root of a minus the cube of b.

Result, $a^2+3c^2b-m(a^{\frac{1}{2}}-b^3)^{\frac{1}{3}}$, or $a^2+3c^2b-m\sqrt[3]{a^{½}-b^3}$.

3. Write three times a into b, plus the binomial a minus b, divided by the sum of a square and b cube.

4. Write the fraction, the product of the sum of a and b into the sum of x and y, divided by the square root of a diminished by the cube root of b.

Result, $\dfrac{(a+b)(x+y)}{\sqrt{a}-\sqrt[3]{b}}$.

5. Write the fraction, a fifth power diminished by 3 times a square b cube, divided by the square root of the binomial x square diminished by y square.

6. Write the square root of the sum of x and y equals c minus the square root of the sum of x and b.

7. Write the fraction, the binomial 3 times x plus 1, divided by 5 times x, minus the fraction 3 times the binomial x minus 1, divided by the binomial $3x$ plus 2, is greater than 9 divided by $11x$.

8. Write the square root of the fraction, b divided by a plus x, plus the square root of the fraction c divided by a minus x, equals the 4th root of the fraction, 4 times the product of b and c, divided by a square minus x square.

9. Write a exponent $\frac{5}{6}$, minus b exponent $-m$. Write the result in three different forms.

10. Write, a exponent $\frac{m}{n}$, is to the binomial b minus x exponent $-\frac{2}{3}$, as 5 times c square plus d, is to the 5th root of x 4th power.

Result, $a^{\frac{m}{n}} : \frac{1}{\sqrt[3]{(b-x)^2}} :: 5c^2+d : \sqrt[5]{x^4}$.

What binomials are there in the last result?

EXERCISES IN READING AND EVALUATING EXPRESSIONS.

Read the following expressions, and find the value of each, when $a = 6$, $b = 5$, $c = 4$, and $d = 1$.

1. $a^2+2ab-c+d$. *Result*, $36+60-4+1$, or 93.

2. $2a^3-3a^2b+c^3$. *Result*, -44.

3. $3(a^2-b^2)-a(c^{\frac{1}{2}}+d)$. *Result*, 15.

4. Between the expression $\frac{4ac}{c^2-d}$ and $\sqrt{5a^2-(c^2+4b)}$, which of the signs, $=$, $>$, or $<$, is correct?

Read and evaluate the following, calling $a = 16$, $b = 10$, $c = 6$, $m = 4$, $x = 5$, $y = 1$.

5. $(b-x)(\sqrt{a}+b)+\sqrt{(a-b)(x+y)}$. *Result*, 76.

6. $\sqrt{c\,(a+b)}-\sqrt[3]{c^2\,(a-b)}$. *Result*, 6.49, nearly.

7. $50ax^{-\frac{m}{2}}+4a^{\frac{8}{3}}-100\left[a-\left(2x+\frac{m}{2}\right)\right]$. *Result*, -112.

8. $3a^{-\frac{1}{2}}+4\left(\frac{2x^2-y^n}{x+2y}+\frac{cm}{b-2y}+\frac{y}{a}\right)$. *Result*, 41.

Suggestion.—Any power of 1 is 1.

9. $3x-\frac{a^2-b^2}{x-y}+\frac{x^2}{b^{-2}}-a^{-\frac{2}{m}}b^2$. *Result*, 2451.

10. Find the value of $a\sqrt{x^2-3a}+x\sqrt{x^2+3a}$, when $x=5$, and $a=8$.

11. Find the value of $a+b\sqrt{(x+y)}-(a-b)\sqrt[3]{(x-y)}$, when $a=10$, $b=8$, $x=12$, and $y=4$.

Suggestion. $\sqrt{(x+y)}$ is equivalent to $\sqrt{x+y}$, the parenthesis and the vinculum having the same signification.

12. If $a=2$, $b=3$, $x=6$, and $y=5$, show that $\sqrt{\{(a+b)\,y\}}+\sqrt[3]{\{(a+x)\,(y-2a)\}}+\sqrt[3]{\{(y-b)^2\,a\}}=9$.

13. With the same values show that $(ay)^3\,(bx+a^2+3)^{-\frac{1}{a}}-\frac{\{b\,(x-y)^{-1}-[(ax)^2-142]\}\,20ay}{(b-a)^m}=0$.

14. Find the value of $\sqrt{10+n}-(10+n)^{\frac{1}{4}}$ if $n=6$.

15. Find the value of $(5m^{\frac{1}{2}}+5\sqrt{x})^{\frac{1}{2}}+\sqrt{m}+x^{\frac{1}{2}}$, if $m=4$, and $x=9$.

Test Questions.—What are the chief points of difference between the Arabic or Decimal notation, and the Literal or Algebraic? What is meant by the terms positive and negative as applied to quantity? What is the meaning of the negative sign when prefixed to an exponent? Read

$$\frac{a^{\frac{1}{2}}-b^{-\frac{2}{3}}}{\sqrt{3x-2y^m}}\propto,\text{ or }=,\text{ or }<,\text{ or }>\sqrt[3]{mx-y^{-n}}:4m^3.$$

SYNOPSIS FOR REVIEW.

NOTATION.

- **DEFINITION.**
- **SYMBOLS OF QUANTITY.**
    - **Arabic.** See Arithmetic.
    - **Literal.**
        - Advantages.
            - *1. More general.*
            - *2. Can trace quantities.*
        - Examples.
        - 1st Law.
            - *Known quantities.*
            - *Unknown quantities.*
            - *Accents, subscripts, and Greek letters.*
            - *Difference between Algebra and Arithmetic.*
        - 2nd Law.
            - *Letters in connection.*
            - *Two points of difference.*
            - *Figures with letters.*
    - **Infinity.** Meaning of.
- **SYMBOLS OF OPERATION AND DEFINITIONS.**
    - $+,\ -,\ \backsim,\ \times,\ \div,\ :,\ \frac{a}{b},\ b)a(c,\ \frac{a|b}{c},\ \sqrt{}.$
    - **Definitions of**
    - **Power,**
    - **Root,**
    - **Exponent,**
    - **Integral Exponent,**
    - **Fractional Exponent,**
    - **Negative Exponent,**
    - **Radical Sign.**
    - How read. *Examples.*
- **SYMBOLS OF RELATION AND DEFINITIONS.**
    - $:,\ \cdot\cdot,\ =,\ ::,\ >\ <,\ \propto.$
    - **Geometrical Ratio.**
    - **Arithmetical Ratio.**
    - **Equality.**
    - **Inequality.**
    - **Variation.**
- **SYMBOLS OF AGGREGATION.**
    - $\overline{\quad},\ (\,),\ [\,],\ \{\,\},\ |.$
    - **Vinculum.**
    - **Vertical Line.**
- **SYMBOLS OF CONTINUATION.** . . . ., - - - - - -.
- **SYMBOLS OF DEDUCTION.** $\therefore,\ \because.$
- **POSITIVE AND NEGATIVE QUANTITIES.**
    - **Definition.**
    - **Two signs of every quantity.**
    - **Plus and minus terms become positive and negative.**
    - **"Less than zero."**
        - Meaning.
        - How negatives increase.
        - How a quantity changes sign
- **FORMS OF EXPRESSION.**
    - **Polynomial.** Term.
    - **Monomial.**
    - **Binomial.**
    - **Trinomial.**
    - **Coefficient.**
    - **Similar Terms.**
    - Illustrations.
- **EXERCISES IN.**
- **EXERCISES IN READING AND EVALUATING EXPRESSIONS.**

ADDITION

SECTION II.

***64.* Addition** is the process of combining several quantities, so that the result shall express the aggregate value in the fewest terms consistent with the notation.

***65.* The Sum or Amount** is the aggregate value of several quantities, expressed in the fewest terms consistent with the notation.

Illustration.—To add 346, 234, and 15, is to find an expression for their aggregate value in the fewest terms consistent with the decimal notation. The *sum* or *amount* is 595, because it is such simplest expression for the aggregate. In like manner the sum of $4ac+5b+2x$, $13ac+2b+8x$, and $12b+9x$, is $17ac+19x+19b$, because it is the simplest expression for the aggregate value consistent with the literal notation.

If the pupil is acquainted with other scales of notation he knows that with radix 100, 595 would be represented by 2 figures, since all numbers less than 100 would be represented by one figure.

***66.* Prop. 1.**—*By Addition similar terms are united into one.*

Demonstration.—Let it be required to add $4ac$, $5ac$, $-2ac$, and $-3ac$. Now $4ac$ is 4 times ac, and $5ac$ is 5 times the same quantity (ac). But 4 times and 5 times the same quantity make 9 times that quantity. Hence, $4ac$ added to $5ac$ make $9ac$. To add $-2ac$ to $9ac$ we have to consider that the negative quantity, $-2ac$, is so opposed in its character to the positive, $9ac$, as to tend to destroy it when combined (added) with it. (As if $9ac$ were property, and $-2ac$ debts.) Therefore, $-2ac$ destroys 2 of the 9 times ac, and gives, when added to it, $7ac$. In like manner $-3ac$ added to $7ac$, gives $4ac$. Thus the four similar terms, $4ac$, $5ac$, $-2ac$, and $-3ac$, have been combined (added) into one term, $4ac$; and it is evident that any other group of similar terms can be treated in the same manner. Q. E. D.

EXAMPLES.

Ex. **1.** Add $13m^2n$, $-10m^2n$, $-6m^2n$, $5m^2n$, and $-4m^2n$.

Model Solution.—Adding together $13m^2n$ and $-10m^2n$, the $-10m^2n$ destroys 10 of the 13 times m^2n and gives $3m^2n$. Adding $3m^2n$ and $-6m^2n$, the $3m^2n$ destroys 3 of the $-6m^2n$ and gives $-3m^2n$. $-3m^2n$ added to $5m^2n$ destroys 3 of the 5 times m^2n and gives $2m^2n$. $2m^2n$ added to $-4m^2n$ destroys 2 of the $-4m^2n$ and gives $-2m^2n$. Hence the sum of $13m^2n$, $-10m^2n$, $-6m^2n$, $5m^2n$, and $-4m^2n$ is $-2m^2n$.

2. Add $18ax^{\frac{1}{2}}$, $-5ax^{\frac{1}{2}}$, $-10ax^{\frac{1}{2}}$, $4ax^{\frac{1}{2}}$, and $-6ax^{\frac{1}{2}}$, explaining as above. *Result*, $ax^{\frac{1}{2}}$.

3. Add $-5c^{\frac{2}{3}}x^2$, $-2c^{\frac{2}{3}}x^2$, $8c^{\frac{2}{3}}x^2$, $3c^{\frac{2}{3}}x^2$, and $-4c^{\frac{2}{3}}x^2$, explaining as before. *Result*, 0.

4. Add $3ax$, $6ax$, $-ax$, $2ax$, $-7ax$, and $5ax$.

5. Add $2by^2$, $-6by^2$, $-by^2$, $8by^2$, $3by^2$, and $-2by^2$.

6. Add $5ax^2$, $-2ax^2$, $3ax^2$, $-9ax^2$, and ax^2.

Sum, $-2ax^2$.

7. Add $5x^{\frac{1}{2}}$, $-6x^{\frac{1}{2}}$, $-10x^{\frac{1}{2}}$, $3x^{\frac{1}{2}}$, and $11x^{\frac{1}{2}}$.

8. Add $-6a^2$, $2a^2$, $-5a^2$, $4a^2$, $-3a^2$, and a^2.

9. Add $-2a\sqrt{x}$, $a\sqrt{x}$, $-3a\sqrt{x}$, $7a\sqrt{x}$, and $-4a\sqrt{x}$.

Sum, $-a\sqrt{x}$.

10. Add $-2am^{\frac{1}{2}}$, $4a\sqrt{m}$, $3am^{\frac{1}{2}}$, and $-a\sqrt{m}$.

Sum, $4a\sqrt{m}$.

11. Add $10a^{\frac{8}{3}}x^{\frac{1}{3}}$, $-4a^{\frac{8}{3}}\sqrt[3]{x}$, $-2a^{\frac{8}{3}}x^{\frac{1}{3}}$, and $4a^{\frac{8}{3}}\sqrt[3]{x}$.

12. Add $1\frac{1}{2}am$, $2\frac{1}{2}am$, $-3am$, and am. *Sum*, $2am$.

13. Add $27a^2$, $-a^2$, $-28a^2$, and $-4a^2$. *Sum*, $-6a^2$.

14. Add $3m^2$, $-\frac{3}{4}m^2$, m^2, and $-\frac{5}{6}m^2$. *Sum*, $2\frac{5}{12}m^2$.

15. Add $7\sqrt{x}$, $-5\sqrt{x}$, $12\sqrt{x}$, and $-3\sqrt{x}$. *Sum*, $11\sqrt{x}$.

16. Add $9b$, $\frac{3}{4}b$, $-\frac{5}{8}b$, $-8b$, and $-\frac{5}{4}b$. *Sum*, $-\frac{1}{8}b$.

67. Cor. 1.—*In adding similar terms, if the terms are all positive, the sum is positive; if all negative, the sum is negative; if some are positive and some negative, the sum takes the sign of that kind (positive or negative) which is the greater.*

Scholium.—The operation of adding positive and negative quantities may look to the pupil like Subtraction. For example, we say +5 and −3 added make +2. This looks like Subtraction, and in one view, it is Subtraction. But why call it Addition? The reason is, because it is simply *putting the quantities together*—aggregating them—not *finding their difference.* Thus, if one boy pulls on his sleigh 5 pounds in one direction, while another boy pulls 3 pounds in the *opposite* direction, the combined (added) effect is 2 pounds in the direction in which the first pulls. If we call the *direction* in which the first pulls, positive, and the *opposite* direction negative, we have +5 and −3 to add. This gives, as illustrated, +2. Hence we see, that the sum of +5 and −3 is +2.

But the *difference* between +5 and −3 is 8, as appears in the following illustration: Suppose one boy is drawing his sleigh forward while another is holding back 3 lbs. If it takes just 10 lbs. to move the sleigh itself, the first boy will have to pull 13 lbs. to get it on. But if instead of *holding back* 3 lbs., the second boy *pushes* 5 lbs., the first boy will only have to pull 5 lbs. Thus it appears, that the *difference* between pushing 5 lbs. (or +5) and *holding* back 3 lbs. (−3) is 8 lbs.

In like manner the sum of $25 of property and $15 of debt, that is the aggregate value when they are combined, is $10. +25 and −15 are +10. But the *difference* between having $25 in pocket, and being $15 in debt, is $40. The difference between +25 and −15 is 40.

17. A thermometer indicated +28° (28° above 0); it then rose 10°, then fell 3°, then rose 2°, and again fell 7°. What was the sum of its movements; or, how did it stand at last?

Model Solution.—Calling upward movement + and downward –, the movements were +10, –3, +2, and –7, the sum of which is +2. Hence it rose 2°. As it originally stood at 28° above 0, and, in the whole, rose 2°, it stands at last 30° above 0.

18. A party are rowing up a stream, and alternately row and rest. During 3 periods of rowing they advance $3mn$, $2mn$, and $6mn$ rods. But during the corresponding periods of resting, they float down $5mn$, mn, and $4mn$ rods. What was the result; did they, on the whole, ascend or descend, and how much? In other words, what is the sum of $+3mn$, $+2mn$, $+6mn$, $-5mn$, $-mn$, and $-4mn$?

Ans. They ascended mn rods. $(+mn.)$

19. A man has a farm worth \$100cd, on which there is a mortgage of \$15cd; he has personal property worth \$8cd, and accounts due him of \$2cd, but owes on account \$5cd, and on note \$7cd. What is the sum of his effects? Or what is the sum of $+100cd$, $-15cd$, $+8cd$, $+2cd$, $-5cd$, and $-7cd$? *Ans.* He is worth \$83cd. $(+83cd.)$

***68.* Cor. 2.**—*The sum of two quantities, the one positive and the other negative, is the numerical difference, with the sign of the greater prefixed.*

***69.* Cor. 3.**—*It appears that addition in mathematics does not always imply increase. Whether a quantity is increased or diminished by adding another to it depends upon the relative nature of the two quantities. If they both tend to the same end, the result is an increase in that direction. If they tend to opposite ends, the result is a diminution of the greater by the less.*

***70.* Prop. 2.**—*Dissimilar terms cannot be united into one by addition, but the operation of adding is represented by writing them in succession, the positive terms being preceded by the + sign and the negative by the – sign.*

Demonstration.—Let it be required to add $+4cy^2$, $+3ab$, $-2xy$, and $-mn$. $4cy^2$ is 4 times cy^2, and $3ab$ is 3 times ab, a different quantity from cy^2; the sum will, therefore, not be 7 times, nor, so far as we can tell, any number of times, cy^2 or ab, or any other quantity, and we can only *represent* the addition thus: $4cy^2+3ab$. In like manner, to add to this sum $-2xy$ we can only represent the addition, as $4cy^2+3ab+(-2xy)$. But since $2xy$ is negative, it tends to destroy the positive quantities and will take out of them $2xy$. Hence the result will be $4cy^2+3ab-2xy$. The effect of $-mn$ will be the same in kind as that of $-2xy$, and hence the total sum will be $4cy^2+3ab-2xy-mn$. As a similar course of reasoning can be applied to any case, the truth of the proposition appears. Q. E. D.

Scholium.—In such an expression as $4cy^2+3ab-2xy-mn$, the $-$ sign before the mn does not signify that it is to be taken from the immediately preceding quantity; nor is this the signification of any of the signs. But the quantities having the $-$ sign are considered as operating to take away so much from *any* which may have the $+$ sign, and *vice versa.*

EXAMPLES.

Ex. **1.** Add together $5ax$, $-10cy$, $8b$, and $-n$.

Model Solution. $5ax$ and $10cy$ being dissimilar will not unite into one term, since one is 5 times ax, and the other is 10 times cy, a *different* quantity; therefore I can only *represent* the addition, as $5ax+(-10cy)$. But the $10cy$ being negative tends to offset positive quantities, and will take out of such its own value. Hence $5ax+(-10cy)$ is $5ax-10cy$. To this adding $8b$, which is positive and hence will go to increase the result, I have $5ax-10cy+8b$. Finally, as n is negative it diminishes the result by its numerical value, and I have for the sum $5ax-10cy+8b-n$.

2. Add together $4am$, $-2c^{\frac{1}{2}}y$, $-8x$, and $5bn$, explaining as above.

3. Add $-2c^2m^{\frac{1}{2}}$, $4cm$, $-6c^{\frac{1}{2}}m^2$, and $10c^2m$, explaining as before.

4. Add $7a^{-2}b$, $-32x^{-\frac{2}{3}}$, $6mn$, and $-5a^2$, explaining as before, and find the numerical value of the result if $a=3$, $b=18$, $x=8$, $m=2$, and $n=5$. *Result*, 21.

71. Cor.—*Adding a negative quantity is the same as subtracting a numerically equal positive quantity; that is,* m+(−n) *is* m−n.

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72. Prob.—To add Polynomials.

Rule.—*I. Write the polynomials so that similar terms shall fall in the same column.*

II. Combine each column into one term, and write the result underneath with its own sign.

The polynomial thus found is the sum sought.

Demonstration.—As the object is to combine the quantities into the fewest terms, it is a matter of convenience to write similar terms in the same column, as such, and only such, can be united into one. (**66, 70.**) Now, since in polynomials the plus and minus terms stand in the same relation to each other as positive and negative quantities (**57**), they may be considered as such, and united by **67.** The partial sums will then be dissimilar terms and will be added by connecting them with their own signs (**70**). Q. E. D.

EXAMPLES.

Ex. 1. Add together $16ac-2m+xy$, $3m-5xy-d-2ac$, $-3xy-4ac-6m$, and $2mn-3ac+8xy$.

Model Solution.—Writing the first polynomial as it stands, I arrange the others so that similar terms shall stand in the same column, for convenience in uniting them. There being no term similar to $+2mn$ I bring it down, and in like

$$\begin{array}{rrrrr}
16ac & -\ 2m & +\ \ xy & & \\
-\ 2ac & +\ 3m & -\ 5xy & -\ d & \\
-\ 4ac & -\ 6m & -\ 3xy & & \\
-\ 3ac & & +\ 8xy & & +\ 2mn \\
\hline
7ac, & -\ 5m, & +\ \ xy, & -\ d, & +\ 2mn \\
\hline
7ac & -\ 5m & +\ \ xy & -\ d & +\ 2mn
\end{array}$$

manner $-d$. $8xy$ and $-3xy$ are $5xy$. $5xy$ and $-5xy$ are 0. 0 and xy, or simply $+xy$, is the sum of the similar terms in xy. Writing this result I pass to the next column. $-6m$ and $+3m$ are $-3m$. $-3m$ and $-2m$ are $-5m$ which being the sum of the similar terms in m, is written down. In like manner the sum of the terms in ac is $7ac$. The partial sums are, therefore, $7ac$, $-5m$, $+xy$, $-d$, and $+2mn$. But these being dissimilar terms are added by connecting them with their own signs (**70**); whence, the sum of the several polynomials is $7ac-5m+xy-d+2mn$.

In like manner solve and explain the following :

2. Add $6x+5ay$, $-3x+2ay$, $x-6ay$, $2x+ay$.
Sum, $6x+2ay$.

3. Add $3ay-7$, $-ay+8$, $2ay-9$, $-3ay-11$, and $10ay-13$.
Sum, $11ay-32$.

4. Add $-3ab+7x$, $3ab-10x$, $3ab-6x$, $-ab+9x$, and $2ab+4x$.
Sum, $4ab+4x$.

5. Add $-6a^2+2b$, $-3b+2a^2$, $-5a^2-8b$, $4a^2-2b$, and $9b-3a^2$.
Sum, $-8a^2-2b$.

6. Add $3a^2b^3-7ab^4+5axy$, $-7a^2b^3-2ab^4-axy$, $ab^4-7axy+8a^2b^3$, $-10ab^4+a^2b^3+3axy$, and $-5a^2b^3+18ab^4$.
Sum, 0.

7. Add $13ax^2-14y^2+3ac^3-mn^2$, $4ax^2+15y^2-3a^3c$, $4y^2-17ax^2+2ac^3+2m^2n+3mn^2$, and $10y^2-a^3c$.
Sum, $15y^2+5ac^3+2mn^2-4a^3c+2m^2n$.

8. Find the sum of $2a^3+4b^2x-c^2x^2$, $2c^2x^2+4a^3-6b^2x$, and $2b^2x-4c^2x^2+2a^3$.
Sum, $8a^3-3c^2x^2$.

9. Find the sum of $8a^2x^2-3xy$, $5ax-5xy$, $9xy-5ax$, $2a^2x^2+xy$, and $5ax-3xy$.
Sum, $10a^2x^2+5ax-xy$.

10. Find the sum of $2bx-12$, $3x^2-2bx$, $5x^2-3\sqrt{x}$, $6\sqrt{x}+12$, x^2+3, and $5x^2-7\sqrt{x}$.
Sum, $14x^2-4\sqrt{x}+3$.

11. What is the sum of $20a^2c^2x+15ah-15a^2c^2x-23ah$?
Ans., $5a^2c^2x-8ah$.

12. What is the sum of $16xy - 4hm + 17xy + 43hm$?

Ans., $33xy + 39hm$.

13. What is the sum of $5c^4 - 49an + 10c^4 + 14an$?

Ans., $15c^4 - 35an$.

14. What is the sum of $10x^{\frac{1}{2}}y^{\frac{1}{4}} - 17sk + 15x^{\frac{1}{2}}y^{\frac{1}{4}} + 5sk - 4x^{\frac{1}{2}}y^{\frac{1}{4}}$?

Ans., $21x^{\frac{1}{2}}y^{\frac{1}{4}} - 12sk$.

Scholium 1.—In practice, the expert will not take the trouble to arrange the polynomials, but will simply select and combine the similar terms, writing each result in the total sum, at once. Thus, in solving *Ex.* 7, when the object is simply to find the sum, and not, as above, to *explain the process*, we proceed as follows: Noting the $13ax^2$, we cast the eye along till we find the similar term, $+4ax^2$, and say "$+17ax^2$;" again, casting the eye along till we find $-17ax^2$, we say "0." Therefore nothing is written in the sum for these terms, as they mutually destroy each other. Again, looking to $-14y^2$, and then on to $15y^2$, we say "y^2," and, passing on to $+4y^2$, say "$5y^2$," and again passing on to $10y^2$ and say "$15y^2$." This being the sum of the terms in y^2, it is then written in the answer. In the same manner the work is carried on to completion; *i. e.*, *only naming results*, and writing them in the total sum.

In this manner, write the answers in the following examples:

15. Add $3x - 5y + 4c$, $2x - 2y - 3c$, and $-x + 3y + c$.

Sum, $4x - 4y + 2c$.

16. Add $11a + 13x - 7d$, $4a - 10x - 2d$, and $-9a - x + 3d$.

Sum, $6a + 2x - 6d$.

17. Add $3ax - 4by + 2mn - 16$, $2by - 5mn + 11$, $3mn - 2ax + 5$, and $-by + mn - ax$.

Sum, $mn - 3by$.

18. Add $6amx - 3b + 4cxy - 2ax^2$, $4b - 3cxy + 5ax^2$, $11cxy - xy - 8amx - 3ax^2$, and $5xy - 6 - b$.

Sum, $12cxy - 2amx + 4xy - 6$.

19. Add $2xy - 2x^2$, $3x^2 + xy$, $x^2 + xy$, and $4x^2 - 3yx$.

20. Add $2ax - 30$, $3x^2 - 2ax$, $5x^2 - 3x^{\frac{1}{2}}$, and $3\sqrt{x} + 10$.
Sum, $8x^2 - 20$.

21. Add $8a^2x^2 - 3ax$, $7ax - 5xy$, $9xy - 5ax$, and $2a^2x^2 + xy$.

22. Add $6ax^2 + 5\sqrt{x}$, $-2ax^2 - 6x^{\frac{1}{2}}$, $3ax^2 - 10x^{\frac{1}{2}}$, $-7ax^2 + 3\sqrt{x}$, and $ax^2 + 11\sqrt{x}$.

23. Add $6xy - 12x^2$, $-4x^2 + 3xy$, $4x^2 - 2xy$, and $-3xy + 4x^2$.

24. Add $4ax - 130 + 3x^{\frac{1}{2}}$, $5x^2 + 3ax + 9x^2$, $7xy - 4\sqrt{x} + 90$, and $\sqrt{x} + 40 - 6x^2$.

25. Add $3a^{-2} + 4bc - e^2 + 10$, $-5a^{-2} + 6bc + 2e^2 - 15$, and $-4a^{-2} - 9bc - 10e^2 + 21$.

26. Add $7a - 5y^3$, $8\sqrt{x} + 2a$, $5y^3 - \sqrt{x}$, and $-9a + 7\sqrt{x}$ together.
Sum, $14\sqrt{x}$.

27. Add $4mn + 3ab - 4c$, $3x - 4ab + 2mn$, and $3m^2 - 4p$ together.
Sum, $6mn - ab - 4c + 3x + 3m^2 - 4p$.

28. Find the sum of $3a^2 + 2ab + 4b^2$, $5a^2 - 8ab + b^2$, $-a^2 + 5ab - b^2$, $18a^2 - 20ab - 19b^2$, and $14a^2 - 3ab + 20b^2$.

29. Find the sum of $4x^3 - 5a^3 - 5ax^2 + 6a^2x$, $6a^3 + 3x^3 + 4ax^2 + 2a^2x$, $-17x^3 + 19ax^2 - 15a^2x$, $13ax^2 - 27a^2x + 18a^3$, $3a^2x - 20a^3 + 12x^3$, and $31a^2x - 2x^3 - 31ax^2 - 7x^3$.
Sum, $-7x^3 - a^3$.

30. Find the sum of $2ab + 12 - x^2y$, $x^{\frac{1}{2}}y + xy + 10$, $3xy^{\frac{1}{2}} + 2x^2y - xy$, $5xy + 11 + x\sqrt{y}$, and $17 - 2x^2y - x^{\frac{1}{2}}y$.
Sum, $2ab + 50 + 5xy - x^2y + 4x\sqrt{y}$.

31. Add $x^{\frac{1}{2}}+ax-ab$, $ab-\sqrt{x}+xy$, $ax+xy-4ab$, $x^{\frac{1}{2}}+\sqrt{x}-x$, and $xy+xy+ax$.

Sum, $2x^{\frac{1}{2}}+3ax-4ab+4xy-x$.

32. Add $7x^{\frac{1}{2}}y-2x\sqrt{y}+7$, $\sqrt{xy}+3xy^{\frac{1}{2}}+2$, $3y\sqrt{x}-\sqrt{yx}-6$, $9y\sqrt{x}-4y^{\frac{1}{2}}x-3$, and $1+7xy^{\frac{1}{2}}-2yx^{\frac{1}{2}}$.

Sum, $18x^{\frac{1}{2}}y+3xy^{\frac{1}{2}}+1$.

Scholium 2.—The object and process of addition, as now explained, will be seen to be identical with the same in Arithmetic, except what grows out of the notation, and the consideration of positive and negative quantities. For example, in the decimal notation let it be required to add 248, 10506, 5003, 81, and 106. The units in the several numbers are similar terms, and hence are combined into one: so also of the tens, and of the hundreds. To make this still more evident, let u stand for units, t for tens, h for hundreds, th for thousands, and $t.th$ for ten thousands. 248 is then $2h+4t+8u$, 10506 is $1t.th+5h+6u$, 5003 is $5th+3u$, 81 is $8t+1u$, and 106 is $1h+6u$. Writing these so that similar terms shall fall in the same column, we have the arrangement in the margin. Whence, adding, we get the sum. The process of carrying has no analogy in the literal notation, since the relative values of the terms are not supposed to be known. Again, there is nothing usually found in the decimal addition like positive and negative quantities. With these two exceptions the processes are essentially the same. The same may be said of addition of compound numbers.

$$\begin{array}{lllll}
 & & 2h & +\ 4t & +\ 8u \\
1t.th & & +\ 5h & & +\ 6u \\
 & 5th & & & +\ 3u \\
 & & & 8t & +\ 1u \\
 & & 1h & & +\ 6u \\
\hline
1t.th & +\ 5th & +\ 9h & +\ 4t & +\ 4u \\
\hline
 & & 15944 & &
\end{array}$$

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73. Prop. 3.—*Literal terms, which are similar only with respect to part of their factors, may be united into one term with a polynomial coefficient.*

Demonstration.—Let it be required to add $5ax$, $-2cx$, and $2mx$. These terms are similar, only with respect to x, and we may say $5a$

times x and $-2c$ times x make $(5a-2c)$ times x, or $(5a-2c)x$. And then, $5a-2c$ times x and $2m$ times x make $(5a-2c+2m)$ times x, or $(5a-2c+2m)x$. Q. E. D.

EXAMPLES.

Ex. 1. Add ax^2, $-bx^2$, $-2cx^2$, and $4mx^2$, with respect to x^2. *Sum*, $(a-b-2c+4m)x^2$.

2. Add $4xy$, $3axy$, $-10mxy$, and cxy, with respect to xy. *Sum*, $(4+3a-10m+c)xy$.

3. Add $amx+2dy$, $2cx-3dy$, and $3dx+5y$, with respect to x and y. *Sum*, $(am+2c+3d)x+(5-d)y$.

4. Find the sum of ax^2+by^2+cxy, and mx^2-ny^2-pxy. *Sum*, $(a+m)x^2+(b-n)y^2+(c-p)xy$.

5. Find the sum of ax^3+bx^2+cx, and $a^2x^3-b^3x^2-c^4x$. *Sum*, $(a+a^2)x^3+(b-b^3)x^2+(c-c^4)x$.

6. Find the sum of $(a-b+c)\sqrt{x}$, $(a+b-c)\sqrt{x}$, and $(b+c-a)\sqrt{x}$. *Sum*, $(a+b+c)\sqrt{x}$.

7. Find the sum of $2ax^{-m}y^{\frac{1}{2}}+3bc-\frac{1}{2}a^{-2}x^{-\frac{2}{3}}+3b$, $3hx^{-m}\sqrt{y}+2bc+\dfrac{3}{2a^2x^{\frac{2}{3}}}-2b$, and $\dfrac{a\sqrt{y}}{x^m}-b+\frac{1}{2}\cdot\dfrac{1}{a^2x^{\frac{2}{3}}}$.

Sum, $3(a+h)\dfrac{\sqrt{y}}{x^m}+5bc+\frac{3}{2}a^{-2}x^{-\frac{2}{3}}$.

8. Add $2x^4y^2+2x^{-m}y^{\frac{3}{2}}-2x^3$, $-5sx^4y^2+2r^2x^{-m}y^{\frac{3}{2}}-a+6x^3$, $3x^4y^2-2b^2x^{-m}y^{\frac{3}{2}}+3a$, and $-2x^3+cy$.

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74. Prop. 4.—*Terms which have a common compound, or polynomial factor, may be regarded as similar and added with respect to that factor.*

Demonstration. $5(x^2-y^2)$, $2(x^2-y^2)$ and $-3(x^2-y^2)$ make, when added with respect to (x^2-y^2), $4(x^2-y^2)$, for they are $5+2-3$, or 4 times the same quantity (x^2-y^2). In a similar manner we may reason on other cases. Q. E. D.

EXAMPLES.

Ex. 1. Find the sum of $\sqrt{x+n}-3x^2$, $5x^2-2\sqrt{x+n}$, $3\sqrt{x+n}-7x^2$, and $10x^2+8\sqrt{x+n}$.

Sum, $10\sqrt{x+n}+5x^2$.

2. Find the sum of $6a-6(a-b)+7$, $3a+12(a-b)-8$, and $2(a-b)-3a-20$. *Sum*, $6a-21+8(a-b)$.

3. Find the sum of $7(m+3)-16(m-3)$, $8(m+3)+7(m-3)$, and $3(m-3)-4(m+3)$.

Sum, $11(m+3)-6(m-3)$.

4. Find the sum of $\frac{1}{2}\sqrt[3]{a-3b}-\frac{1}{4}\sqrt[3]{a-3b}+\frac{1}{8}\sqrt[3]{a-3b}-\frac{1}{16}\sqrt[3]{a-3b}$. *Sum*, $\frac{5}{16}\sqrt[3]{a-3b}$.

5. Add $3(a-c)(x+y^2)^{-\frac{1}{2}}$, $\dfrac{a}{(x+y^2)^{\frac{1}{2}}}$, $\dfrac{c}{\sqrt{x+y^2}}$, and $5(a+c)(x+y^2)^{-\frac{1}{2}}$. *Sum*, $\dfrac{3(3a+c)}{\sqrt{x+y^2}}$.

6. Find the sum of $a(a+b)+3\sqrt{a-x}$, $-4a(a+b)+7a(a-x)^{\frac{1}{2}}$, $-6a\sqrt{a-x}+11a(a+b)$, $-2a(a+b)-2(a-x)^{\frac{1}{2}}$, and $5a(a+b)+14\sqrt{a-x}$.

7. Add $a\sqrt{x-y}+bxy+c(a+x)^2$, $-bxy+(a+c)(a+x)^2+(x-y)^{\frac{1}{2}}$, $2bxy+(a-1)\sqrt{x-y}-a(a+x)^2$.

Sum, $2a(x-y)^{\frac{1}{2}}+2bxy+2c(a+x)^2$.

8. Show that $\sqrt{x}+by+ax-z+amy+c\sqrt{x}+dz+y$ $=(am+b+1)y+(c+1)x^{\frac{1}{2}}+(d-1)z+ax$.

Test Questions.—Does addition always imply an increase? When does it not? When does it? What is addition? How are similar terms added? How are dissimilar added? Give the Rule for adding polynomials and demonstrate it?

SYNOPSIS.

ADDITION.

- **DEFS.**
  - Addition.
  - Sum, or Amount.
- **PROPS.**
  - 1. Similar Terms. DEM.
    - Cor. 1.—*Sign of sum.*
    - Cor. 2.—*Sum of Positive and Negative.*
    - Cor. 3—*Addition not always increase.*
  - 2. Dissimilar Terms. DEM.
    - Sch.—*Sign of term.*
    - Cor.—*Addition of Negative = Subtraction of Positive.*
- **PROB.** To add Polynomials. RULE. DEM.
  - Sch. 1.—*Practical method.*
  - Sch. 2.—*Same as in Decimal Notation.*
- **PROPS.**
  - 3. Terms Partially Similar.
  - 4. Compound Similar Terms.

Subtraction

SECTION III.

75. **Subtraction** is, primarily, the process of taking a *less* quantity from a greater.

In an enlarged sense, Subtraction comes to mean taking one quantity from another irrespective of their magnitudes.

Subtraction also comprehends all processes of finding the difference between quantities.

The terms Minuend, Subtrahend, and Remainder, are used as in Arithmetic.

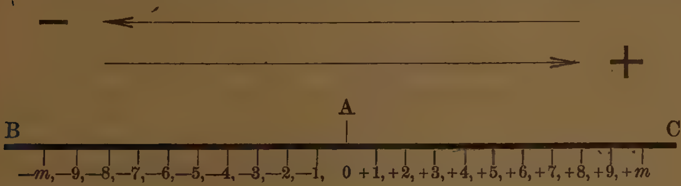
76. The Difference between two quantities is, in its primary signification, the number of units which lie between them; or, *it is what must be added to one in order to produce the other.*

When it is required to take one quantity from another, *the difference is what must be added to the Subtrahend in order to produce the Minuend.*

Scholium.—The most comprehensive and fundamental notion of *difference* is this: Having reached any specified point in a scale of numbers, or in estimating magnitude, in what direction, and *how far* must we pass to reach another specified point in the scale of numbers, or in the value of the magnitude.

Illustration.—When we ask, "What is the difference between 3 and 8?" we ordinarily mean, "Over how many units must we pass in reckoning from 3 to 8?" That is, "How many units added to 3 will make 8?"

Let us use the following device to illustrate the whole subject. Consider A the zero point on the line BC. Call distances to the right of A positive (+), and distances to the left negative (−). Also call reckoning towards the right positive and towards the left negative, from any point on the scale.


−
+
A
B
C
−m, −9, −8, −7, −6, −5, −4, −3, −2, −1, 0 +1, +2, +3, +4, +5, +6, +7, +8, +9, +m

1st. The difference between 3 and 8, means either, *how far, and in what direction* must we go to pass from 3 to 8 or to pass from 8 to 3? In the first case we pass 5 to the right, and say 3 from 8 is +5, understanding that 3 and 8 are both positive. But to pass from 8 to 3, we pass 5 towards the left, and hence say 8 from 3 gives −5. These two operations are represented thus: $8-3=5$, and $3-8=-5$.

2nd. The difference between −3 and −8 means either, *how far, and in what direction* must we pass to go from −3 to −8, or from −8 to −3? To pass from −3 to −8, we pass 5 towards the left, and hence say -3 from -8 gives -5. That is $-8-(-3) = -5$. But to pass from -8 to -3, we pass 5 to the right, hence $-3-(-8) = +5$.

3d. The difference between -3 and $+8$ means either, *how far and in what direction* must we pass to go from -3 to $+8$, or from $+8$ to -3? In the first case we get $+8-(-3) = +11$, since we pass 11 to the right. In the second case we get $-3-(+8) = -11$, since to go from $+8$ to -3 we pass 11 to the left.

In each and all of these cases, the question is, "What must be added to the quantity conceived as the Subtrahend, in order to produce the Minuend?" considering, in each instance, the number of units between the two given terms as the numerical value of the difference, and its sign as determined by the fact as to whether we reckon to the right (up the scale), or to the left (down the scale), in passing from the Subtrahend to the Minuend.

77. Prob.—To perform Subtraction.

Rule.—*Change the signs of each term in the subtrahend from + to −, or from − to +, or conceive them to be changed, and add the result to the minuend.*

Demonstration.—Since the difference sought is what must be added to the subtrahend to produce the minuend, we may consider this difference as made up of two parts, one the subtrahend with its signs changed, and the other the minuend. When the sum of these two parts is added to the subtrahend, it is evident that the first part will destroy the subtrahend, and the other part, or minuend, will be left.

Thus, to perform the example:

$$\begin{array}{l} \text{From} \\ \text{Take} \\ \text{Subtrahend with signs changed,} \\ \text{Minuend,} \\ \text{Difference,} \end{array} \qquad \begin{array}{r} 5ax-6b-3d-\ 4m \\ \left.\begin{array}{r} \underline{2ax+2b-5d+\ 8m} \\ -2ax-2b+5d-\ 8m \\ \underline{5ax-6b-3d-\ 4m} \end{array}\right\} \\ 3ax-8b+2d-12m \end{array}$$

If the three quantities included in the brace are added together, the sum will evidently be the minuend. If, therefore, we add the second and third of them (that is the subtrahend, with its signs changed, and the minuend) together, the sum will be what is necessary to be added to the subtrahend to produce the minuend, and hence is the difference sought. Q. E. D.

EXAMPLES.

Ex. **1.** From $4ab+3c^2-xy$ subtract $3xy-2z+2ab-c^2$.

Model Solution.—Writing the subtrahend under the minuend so that similar terms shall fall under each other, for the convenience of combination, I have

$$\begin{array}{l} 4ab + 3c^2 - xy \\ \underline{2ab - c^2 + 3xy - 2z} \\ 2ab + 4c^2 - 4xy + 2z \end{array}$$

The difference sought may be considered as consisting of two parts: 1st, the subtrahend with its signs changed, and 2nd, the minuend itself. The sum of these two parts is the difference, since it is what is necessary to be added to the subtrahend to produce the minuend. Therefore conceiving the signs of the subtrahend to be changed, and adding, I have $2ab+4c^2-4xy+2z$, as the difference sought.

A more detailed Explanation is as follows: The first question is, What must be added to $-2z$ to produce the corresponding term in the minuend? This is evidently $+2z$, as there is no corresponding term in the minuend, and I have only to write a term in the difference which will destroy $-2z$ when added to it. Passing to the next term, I inquire, What must be added to $+3xy$ to produce $-xy$? First, I must add $-3xy$ (the term with its sign changed) in order to destroy the $+3xy$, and then I must add $-xy$ in order to make the $-xy$ of the minuend. So in all I must add $-3xy$ and $-xy$, or $-4xy$. $-4xy$ is, therefore, the difference sought. Passing to the next term, What must be added to $-c^2$ to make $+3c^2$? I must add $+c^2$ (to destroy $-c^2$) and $+3c^2$ (to make up the required term in the minuend), or in all $+c^2$ and $+3c^2$, or $+4c^2$; which is the difference. Finally, the term to be added to $2ab$ in order to make $4ab$ is composed of the two parts $-2ab$ and $+4ab$, which make $2ab$.

It thus appears that $2ab+4c^2-4xy+2z$ is the difference, since *it is what must be added to the subtrahend to produce the minuend.*

2. From $4x-3b^2+2xy-11$ take $2x+b^2+2xy-5$.

Rem., $2x-4b^2-6$.

3. From $15ax+2b^2y-6a^2x^2$ take $-5ax-4b^2y-3a^2x^2$.

Rem., $20ax+6b^2y-3a^2x^2$.

4. From $8a^{\frac{1}{2}}b^{-2}-3xy+15-2\sqrt{xy}$ take $8+10xy-3a^{\frac{1}{2}}b^{-2}+2\sqrt{xy}$.

Rem., $11a^{\frac{1}{2}}b^{-2}-13xy+7-4\sqrt{xy}$.

5. From $4a^2x^{\frac{1}{2}}-3c+12b^{\frac{2}{3}}y^{\frac{1}{2}}$ take $8b^{\frac{2}{3}}y^{\frac{1}{2}}+4a^2x^{\frac{1}{2}}-2d$.

Rem., $4b^{\frac{2}{3}}y^{\frac{1}{2}}-3c+2d$.

6. From $a^2-2ab+b^2$ take $a^2+2ab+b^2$.

Rem., $-4ab$.

7. From a^2-b^2 take $a^2-2ab-b^2$. *Rem.*, $2ab$.

8. From $24xy^2-14my+18x^2y^2-14+27xz^2$ take $17xy^2-10my-4x^2y^2+20xz^2-8$.

9. From $17pmx^2-18n^3+19m^4-24$ take $7pmx^2-4n^3+10m^4-17$.

10. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$.

11. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$.

12. From $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ take $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$.

13. From $6\sqrt{x+y}+4a^2x^2$ take $3\,(x+y)^{\frac{1}{2}}-4a^2x^2$.

Diff., $3\sqrt{x+y}+8a^2x^2$.

Suggestion.—Regard $\sqrt{x+y}$ as one quantity, and observe that $3\,(x+y)^{\frac{1}{2}}$ is the same as $3\sqrt{x+y}$.

14. From $6x-3\sqrt{xy}+17\,(a+b)$ take $2x+7\sqrt{xy}+4\,(a+b)$.

Diff., $4x-10\sqrt{xy}+13(a+b)$.

15. From $10-a+b-c+x$ take $a+b-c-x-90$.

16. From $6m+\frac{1}{2}n-\frac{2}{3}a-1$ take $2m-\frac{1}{2}n+\frac{1}{3}a-2$.

Diff., $4m+n-a+1$.

Scholium 1.—When several polynomials are to be combined, some by addition and some by subtraction, it will be found expedient to write them so that similar terms will fall under each other, writing the several subtrahends with their signs changed, and then add the quantities as they stand.

17. From the sum of $2ay^2-2b+3ax$, $2ay^2+b-ax$, and $3b-ay^2+3-y$, subtract $3ay^2+4b-ax+y$.

Result, $-2b+3ax+3-2y$.

18. From the sum of $3a^2+xy^2-2by$, $5a^2+3xy^2-3by$, and $3xy^2+4a^2+by$, subtract $2a^2-xy^2-5by+5$.

Result, $10a^2+8xy^2+by-5$.

19. From the sum of $a^2b^2-3y^2+5xy$, $3a^2b^2+3y^2-2xy$, and $5y^2+3a^2b^2-xy+6$, subtract $a^2b^2+xy-y^2$.

Result, $6a^2b^2+6y^2+xy+9$.

20. To $5ax^2-7cb+8m^{\frac{1}{2}}-2c^{-n}$, add $3cb-4c^{-n}+2ax^2$, then subtract $10m^{\frac{1}{2}}-5xy+3cb-12c^{-n}$, add $6m^{\frac{1}{2}}-11cb+3n-4ax^2$, subtract $-16ax^2-2m^{\frac{1}{2}}+4cb$, add $5m^{\frac{1}{2}}+6c^{-n}$, and subtract $4xy-2n$.

Result, $19ax^2-22cb+11m^{\frac{1}{2}}+12c^{-n}+xy+5n$.

21. To $7cy^{-\frac{2}{3}}+8ax-5b$, add $4b-2cy^{-\frac{2}{3}}+m$, then subtract $5ax-4m+3$ and $-3ax+5cy^{-\frac{2}{3}}-6$, add $10ax-2b+8m-3$, and subtract $3m-10cy^{-\frac{2}{3}}-2m$.

Result, $10cy^{-\frac{2}{3}}+16ax-3b+12m$.

Scholium 2.—The proficient solves such examples as the above, and, in fact, all kindred ones, without re-writing the quantities. Thus, to obtain the result in *Ex.* 20, he looks at $5ax^2$, casts his eye along till he sees $2ax^2$, says $7ax^2$; then looks forward to $-4ax^2$, and says $3ax^2$; then notices $-16ax^2$ in a *subtrahend*, and hence *thinks of it* as $+16ax^2$, and says $19ax^2$. Again taking up $-7cb$, he looks along noticing the similar terms and says, mentally, $-4cb$, $-7cb$, $-18cb$, $-22cb$. Again, he takes up $8m^{\frac{1}{2}}$, and running through with the similar terms, says, $-2m^{\frac{1}{2}}$, $4m^{\frac{1}{2}}$, $6m^{\frac{1}{2}}$, $11m^{\frac{1}{2}}$. And so for all the other terms.

This is the practical way of solving such examples; and the pupil should exercise himself till he can write out the result at once. He can go over the preceding examples thus, and should also solve, without writing, those which follow.

22. From $13x^2-2ax-9b^2$ take $5x^2-7ax-b^2$.

23. From $20ax-\sqrt{x}+3d$ take $4ax+5x^{\frac{1}{2}}-d$.

24. From $5ab+2b^2-c+bc-b$ take $b^2-2ab+bc$.

25. From $ax^{-3}-ax^{-2}+cx-d$ take $ax^3-ax^{-2}-ex-2d$.

Diff., $a(x^{-3}-x^3)+(c+e)x+d$.

Suggestion.—This would at first become $ax^{-3}-ax^3+cx+ex+d$. But this may evidently be written as above.

26. From $x^2y-3\sqrt{xy}-6ay$ take $3x^2y+3(xy)^{\frac{1}{2}}-4ay$.

27. From the sum of $4ax-150+4x^{\frac{1}{2}}$, $5x^2+3ax+10\sqrt{x}$, and $90-2ax-12\sqrt{x}$; take the sum of $2ax-80+7x^2$, $7x^{\frac{1}{2}}-8ax-70$, and $30-4x^{\frac{1}{2}}$, $-2x^2+4a^2x^2$.

Result, $11ax+60-x^{\frac{1}{2}}-4a^2x^2$.

78. Cor. 1.—*When a parenthesis, or any symbol of like signification* (**50**), *occurs in a polynomial, preceded by a — sign, and the parenthesis or equivalent symbol is removed, the signs of all the terms which were within must be changed, since the sign — indicates that the quantity within the parenthesis is a subtrahend.*

28. Remove the parenthesis from the polynomial $3a^{\frac{1}{2}}x+2b^2y^2-(5a^{\frac{1}{2}}x-2m^2z+8b^2y^2)$ and represent the result in its simplest form. Do the work mentally, writing only the result.

Result, $2m^2z-2a^{\frac{1}{2}}x-6b^2y^2$.

29. Remove the parenthesis from $5a-4b+3c-(-3a+2b-c)$.

Result, $8a-6b+4c$.

30. Remove the parenthesis from $4a-5x-(a-4x)+(x-8a)$. *Result*, $-5a$.

Queries.—In *Ex.* 28 is the sign of $5a^{\frac{1}{2}}x$ changed? What is its sign as it stands in the parenthesis? Is the — sign which appears in the result before $5a^{\frac{1}{2}}x$, the same as the one before the parenthesis in the example? No. What became of that before the parenthesis? *Ans.* The operation which it indicated having been performed, it was dropped. In *Ex.* 30, why are not the signs of the terms in the last parenthesis changed?

79. Cor. 2.—*A parenthesis preceded by the — sign can be placed around any number of terms by changing the signs of all the terms. The reason of this is evident, since by removing the parenthesis according to the preceding corollary, the expression would return to its original form.*

31. Introduce within a parenthesis the 3d, 4th and 5th terms of the following expression: $6ax-2cd-8m+5x-2y+x-4a$.

Result, $6ax-2cd-(8m-5x+2y)+x-4a$.

32. Introduce within a parenthesis the last three terms of $4xy+2cb-8x-5+2b$.

Result, $4xy+2cb-(8x+5-2b)$.

33. Include in brackets the 3d, 4th, and 5th terms of $5ax-2x^2+3x-12ay+15$. Also the 4th and 5th. Also the 2d and 3d.

Form of the first and last, $5ax-2x^2+(3x-12ay+15)$, $5ax-(2x^2-3x)-12ay+15$.

Query.—In the last is the sign of $2x^2$ changed?

34. Prove that $(3x-6y)+(4y-4x)+2(x+2y)=x+2y$.

35. Prove that $\frac{1}{2}(a+b-c)+\frac{1}{2}(b+c-a)=b$.

36. Prove that $6a-4b-2(a+b)=2(2a-3b)$.

37. Prove that $x^2+6x^2y+3-(x^2+4x^2y+1)=2(x^2y+1)$.

***38.** Prove that $\frac{1}{3}(a-5b+\frac{1}{2}c)+\frac{1}{4}(5b-2a+\frac{1}{3}c)=\frac{1}{4}c-\frac{5}{12}b-\frac{1}{6}a$.

39. Prove that $\frac{1}{3}(a^2-a+1)+\frac{1}{4}(2a^2-a+2)=\frac{1}{12}(10a^2-7a+10)$.

40. Prove that $\frac{1}{2}a+\frac{1}{2}b-(\frac{1}{2}a-\frac{1}{2}b)=b$.

41. Prove that $\frac{1}{3}(9-15x)-\frac{1}{2}(12-20x)-\frac{1}{5}(45x-20)=1-4x$.

42. Prove that $\frac{1}{2}(a-\frac{1}{2}b+\frac{1}{3}c)+\frac{1}{4}(a-\frac{1}{5}b+\frac{1}{6}c)=\frac{1}{120}(90a-36b+25c)$.

80. Cor. 3.—*When several parentheses occur, included the one within the other, begin the removal with the inside one.*

43. Remove the parentheses and other marks of aggregation from $4a-\{-[c-d+(4x^2-1)-xy]-3y\}$.

Result, $4a+c-d+4x^2-1-xy+3y$.

44. Show that $a-[b-\{c-(d-\overline{e-f})\}]$

$$
\begin{aligned}
&=a-[b-\{c-(d-e+f)\}]\\
&=a-[b-\{c-d+e-f\}]\\
&=a-[b-c+d-e+f]\\
&=a-b+c-d+e-f.
\end{aligned}
$$

45. Remove the marks of aggregation from the expression $7a-\{3a-[4a-(5a-2a)]\}$, and *afterwards* reduce the result to its simplest form. Also combine and remove at the same time. *Result,* $5a$.

46. Remove the marks of aggregation from $a+2b-\{6a-[3b+(8x-\overline{2+by}-x)+4a]-3b\}$.

Result, $8b-a+7x-by-2$.

Scholium 3.—Terms having common literal factors may be regarded as similar with respect to these, and treated accordingly, the other factors in each term being regarded as coefficients.

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* This, and the examples which follow in this section, may be omitted and taken in the review after Fractions, if thought best by the teacher.

47. From $ax+by-cz$ take $my-nx+2z$.
Result, $(a+n)x+(b-m)y-(c+2)z$.

48. From $ax^2+bxy+cy^2$ take $dx^2-hxy+ky^2$.
Result, $(a-d)x^2+(b+h)xy+(c-k)y^2$.

49. From $\sqrt{x+y}+3ax-12$ take $4(x+y)^{\frac{1}{2}}+b-2ax$.
Result, $5ax-3(x+y)^{\frac{1}{2}}-12-b$.

50. From $a^2x^2-4axy+4a^2x^2y^2$ take $c^2x^2-8cxy+6x^2y^2$.
Result, $(a^2-c^2)x^2-(4a-8c)xy+(4a^2-6)x^2y^2$.

Show that the second term in the last result may be $+(8c-4a)$.

51. From $\sqrt{x^2-y^2}-2(a+x)^{\frac{1}{2}}+3$ take $-3\sqrt{a+x}$ $+4(x^2-y^2)^{\frac{1}{2}}-1$. *Result*, $(a+x)^{\frac{1}{2}}-3\sqrt{x^2-y^2}+4$.

SYNOPSIS FOR REVIEW.

- **SUBTRACTION.**
  - **DEFINITIONS.**
    - **Subtraction.**
    - **Subtrahend.**
    - **Minuend.**
    - **Difference.**
      - SCHOLIUM.
      - ILLUSTRATION. DIAGRAM.
  - **GENERAL PROBLEM.**
    - **Rule.** DEM.
      - Sch. 1.—Both add. and sub.
      - Sch. 2.—Practical method.
  - **BRACKETS.**
    - **Cor. 1.—To remove.**
    - **Cor. 2.—To introduce.**
    - **Cor. 3.—Several.**
  - **TERMS PARTIALLY SIMILAR.** **Sch. 3.**

Test Questions.—What *two* answers can you give to the question, "What is the difference between 10 and 6?" Why do you change the signs of the subtrahend in subtracting? Why do you add the subtrahend, with signs changed, to the minuend? When do you change the signs in removing a parenthesis? Why? What becomes of the sign before the brackets? In removing a parenthesis preceded by a — sign, is the sign of the first term changed as well as the others? State in the briefest manner the theory of subtraction? *Reply.* Subtraction is finding the difference between quantities, that is, finding what must be added to one quantity to produce the other. This difference may always be considered as consisting of two parts, one of which destroys the subtrahend, and the other part is the minuend itself. Hence, to perform subtraction, we change the signs of the subtrahend to get that part of the difference which destroys the subtrahend, and add this result to the minuend, which is the other part of the difference.

Multiplication

SECTION IV.

81. **Multiplication** is the process of finding the simplest expression for a quantity which shall be as many times a given quantity, or such a part of that quantity, as is represented by a specified number.

The quantity to be multiplied is called the *Multiplicand.* The number by which we multiply is called the *Multiplier.* Taken together, the multiplier and multiplicand are called *Factors.* The result is the *Product.*

82. **Cor. 1.**—*The multiplier must always be conceived as an abstract number, since it shows* HOW MANY TIMES *the multiplicand is to be taken.*

Thus, to propose to multiply $12 by $5 is absurd. We can understand that 5 times $12 is $60; but what is meant by 5 DOLLARS times?

83. **Cor. 2.**—*The product is always of the same kind as the multiplicand.*

84. **Scholium.**—It is frequently convenient in practice to speak of the multiplier as positive or negative, although, literally understood, this is a contradiction of Cor. 1, which requires the multiplier to be conceived as mere number. In a strict analysis, the multiplier in such cases is to be considered, first, without reference to its sign, *i. e.*, as abstract, and then the sign is to be interpreted as indicating what is to be done with the product, when it is taken in connection with other quantities.

---

85. Prop. 1.—*The product of several factors is the same in whatever order they are taken.*

Demonstration.—1st. $a \times b$, is a taken b times, or $a+a+a+a+a$ - - - - - to b terms. Now, if we take 1 unit from each term (each a), we shall get b units; and this process can be repeated a times, giving a times b, or $b \times a$. $\therefore$ $a \times b = b \times a$.

2nd. When there are more than two factors, as abc. We have shown that $ab = ba$. Now call this product m, whence $abc = mc$. But by part 1st, $mc = cm$. $\therefore$ $abc = bac = cab = cba$. In like manner we may show that the product of any number of factors is the same in whatever order they are taken. Q. E. D.

Scholium.—If the multiplicand is concrete, the reasoning is still the same. Thus $\$a \times b = \$a + \$a + \$a + \$a$, - - - - - - - - etc. to b terms. Now take \$1 from each of the terms of $\$a$ each, and we have $\$b$; and this process can be repeated a times, giving $\$b \times a$. $\therefore$ $\$a \times b = \$b \times a$. Notice that in each case the multiplier is abstract.

86. Prop. 2.—*When two factors have the same sign their product is positive; when they have different signs their product is negative.*

Demonstration.—1st. Let the factors be $+a$ and $+b$. Considering a as the multiplier, we are to take $+b$, a times, which gives $+ab$, a being considered as abstract in the operation, and the product, $+ab$, being of the same kind as the multiplicand; that is, positive. Now, when the product, $+ab$, is taken in connection with other quantities, the sign $+$ of the multiplier, a, shows that it is to be *added*; that is, written with its sign unchanged. $\therefore$ $(+b) \times (+a) = +ab$.

2nd. Let the factors be $-a$ and $-b$. Considering a as the multiplier, we are to take $-b$, a times, which gives $-ab$, a being considered as abstract in the operation, and the product, $-ab$, being of the same kind as the multiplicand; that is, negative. Now, when this product, $-ab$, is taken in connection with other quantities, the sign $-$ of the multiplier shows that it is to be subtracted; that is, written with its sign changed. $\therefore\ (-b) \times (-a) = +ab$.

3d. Let the factors be $-a$ and $+b$. Considering a as the multiplier, we are to take $+b$, a times, which gives $+ab$, a being considered as abstract in the operation, and the product, $+ab$, being of the same kind as the multiplicand; that is, positive. Now, when this product, $+ab$, is taken in connection with other quantities, the sign $-$ of the multiplier shows that it is to be subtracted; that is, written with its sign changed. $\therefore\ (+b) \times (-a) = -ab$.

4th. Let the factors be $+a$ and $-b$. Considering a as the multiplier, we are to take $-b$, a times, which gives $-ab$, a being considered as abstract in the operation, and the product, $-ab$, being of the same kind as the multiplicand; that is, negative. Now, when this product, $-ab$, is taken in connection with other quantities, the sign $+$ of the multiplier shows that it is to be added; that is, written with its own sign. $\therefore\ (-b) \times (+a) = -ab$. Q. E. D.

87. Cor. 1.—*The product of any number of positive factors is positive.*

Thus $(+a) \times (+b) \times (+c) \times (+d) = abcd$, since $(+a) \times (+b) = +ab$, which, in turn multiplied by $+c$, gives $+abc$, etc.

88. Cor. 2.—*The product of an even number of negative factors is positive; since we can multiply them two and two, thus obtaining positive products, which positive products multiplied together make the complete product positive.*

Thus $(-a) \times (-b) \times (-c) \times (-d) \times (-e) \times (-f) = (+ab) \times (+cd) \times (+ef) = +abcdef$, or $abcdef$.

89. Cor. 3.—*The product of an odd number of negative factors is negative; since, by the last corollary, the product of all but one (an even number) of such factors is positive, and then this multiplied by the remaining negative factor gives $(+) \times (-)$, and hence is negative.*

---

90. Prop. 3.—*The product of two or more factors consisting of the same quantity affected with exponents, is the common quantity with an exponent* *equal to the sum of the exponents of the factors.* That is, $a^m \times a^n = a^{m+n}$; or $a^m \cdot a^n \cdot a^s = a^{m+n+s}$, etc., whether the exponents are integral or fractional, positive or negative.

Demonstration.—1st. *When the exponents are positive integers.* Let it be required to multiply a^3 by a^2. $a^3 = aaa$, and $a^2 = aa$. $\therefore$ $a^3 \times a^2 = aaa \cdot aa = a^5$. That is, there are *three* factors each a, in a^3, and *two* like factors in a^2; and, as the product consists of all the factors in both multiplier and multiplicand, it will contain *five* factors each a, and hence is a^5.

In general: *To multiply* a^m by a^n *and* a^s. $a^m = aaa$ - - - - to m factors, $a^n = aaaaa$ - - - - to n factors, and $a^s = aaaaa$ - - - - to s factors. Hence the product, being composed of all the factors in the quantities to be multiplied together, contains $m+n+s$ factors each a, which is represented a^{m+n+s}. Since it is evident that this reasoning can be extended to any number of factors; the proposition is proved in the case of positive integral exponents.

2nd. *When the exponents are positive fractions.* Let it be required to multiply $64^{\frac{2}{3}}$ by $64^{\frac{5}{6}}$. Now $64^{\frac{2}{3}} = 4 \cdot 4$, *i. e.*, 2 of the 3 equal factors which make 64. In like manner $64^{\frac{5}{6}}$ is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. And since $4 \cdot 4$ is $2 \cdot 2 \cdot 2 \cdot 2$, $64^{\frac{2}{3}} \times 64^{\frac{5}{6}} = 2 \cdot 2 \cdot 2 \cdot 2 \times 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^9$, or 9 of the 6 equal factors into which 64 is resolvable, and may be represented $64^{\frac{9}{6}}$ $\therefore$ $64^{\frac{2}{3}} \times 64^{\frac{5}{6}} = 2^9 = 64^{\frac{9}{6}} = 64^{\frac{2}{3}+\frac{5}{6}}$.

In general, let $a^{\frac{m}{n}}$ be multiplied by $a^{\frac{c}{b}}$. $a^{\frac{m}{n}}$ means m of the n equal factors composing a. Now, if each of these n factors be resolved into b factors, a will be resolved into bn factors, and to make the quantity $a^{\frac{m}{n}}$ we shall have to take bm instead of m factors. Hence $a^{\frac{m}{n}} = a^{\frac{bm}{bn}}$. In like manner $a^{\frac{c}{b}}$ may be shown equal to $a^{\frac{cn}{bn}}$; whence $a^{\frac{m}{n}} \times a^{\frac{c}{b}} = a^{\frac{bm}{bn}} \times a^{\frac{cn}{bn}}$. This now signifies that a is to be resolved into bn factors, and $bm+cm$ of them taken to form the product. $\therefore$ $a^{\frac{m}{n}} \times a^{\frac{c}{b}} = a^{\frac{bm}{bn}} \times a^{\frac{cn}{bn}} = a^{\frac{bm+cn}{bn}}$, or $a^{\frac{m}{n}+\frac{c}{b}}$. Finally, as it is evident that this reasoning can be extended to any number of factors, the proposition is proved in the case of positive fractional exponents.

3d. *When the exponents are negative.* Let it be required to multiply 2^{-3} by 2^{-2}. 2^{-3} is $\frac{1}{2^3}$, and 2^{-2} is $\frac{1}{2^2}$ $\therefore 2^{-3} \times 2^{-2} = \frac{1}{2^3} \times \frac{1}{2^2}$ $= \frac{1}{2^5}$, or 2^{-5}.

In general, $a^{-m} \times a^{-n} = a^{-m-n}$. For $a^{-m} = \frac{1}{a^m}$, and $a^{-n} = \frac{1}{a^n}$ (**43**). Whence $a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^m \times a^n} = \frac{1}{a^{m+n}}$ by the preceding parts of this demonstration; and by (**43**) $\frac{1}{a^{m+n}} = a^{-m-n}$. This reasoning may also be extended to any number of factors. Q. E. D.

EXAMPLES.

Ex. 1. Prove as above that $81^{\frac{3}{2}} \times 81^{\frac{3}{4}} = 81^{\frac{18}{8}}$ and that $81^{\frac{18}{8}} = 81^{\frac{9}{4}}$.

2. Prove that $m^a \times m^b = m^{a+b}$.

3. Prove that $16^{-\frac{3}{4}} \times 16^{-\frac{1}{2}} = 16^{-\frac{5}{4}}$.

4. Prove that $25^{-\frac{1}{2}} \times 25^{\frac{1}{2}}$ is 1.

5. Prove that $a^{-2} \times a^3$ is a.

Scholium.—The student must be careful to notice the difference between the signification of a fraction *used as an exponent*, and its common signification. Thus $\frac{2}{3}$ *used as an exponent* signifies that a number is resolved into 3 equal *factors*, and the *product* of 2 of them taken; whereas $\frac{2}{3}$ *used as a common fraction* signifies that a quantity is to be separated into 3 equal *parts*, and the *sum* of two of them taken.

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91. Prob.—To multiply Monomials.

Rule.—*Multiply the numerical coefficients, and to this product affix the letters of all the factors, affecting each with an exponent equal to the sum of all the exponents of that letter in all the factors.*

The sign of the product will be + except when there is an odd number of negative factors; in which case it will be −.

Demonstration.—This rule is but an application of the preceding principles. Since the product is composed of all the factors of the given factors, and the order of arrangement of the factors in the product does not affect its value, we can write the product, putting the continued product of the numerical factors first, and then grouping the literal factors so that like letters shall come together. Finally, performing the operations indicated, by multiplying the numerical factors as in the decimal notation, and the like literal factors by adding the exponents, the product is completed.

EXAMPLES.

Ex. **1.** Multiply together $3a^2bx$, $2cb^3y$, and $5ac^2x$.

Model Solution.—Since the product must contain all the factors of the given factors, and the order of arrangement is immaterial (**85**), $3a^2bx \times 2cb^3y \times 5ac^2x = 3 \cdot 2 \cdot 5 \times a^2a \times bb^3 \times cc^2 \times xx \times y$, which, by performing the operations indicated, becomes $30a^3b^4c^3x^2y$.

2. Multiply together $3abxy$, $2a^4bx^3$, $10cx^2$, $4y^2$, and a.
Prod., $240a^6b^2cx^6y^3$.

3. Multiply together $5ax^2$, $-2by$, $-3a^2c$, ax^3, and $-2y^2$.
Prod., $-60a^4bcx^5y^3$.

4. Multiply together mx^n, $2m^2x^3$, $-3ax^2$, $-5m^s$, and $4a^3x^r$.
Prod., $120a^4m^{3+s}x^{5+r+n}$.

5. What is the product of $-2c^md$, $-10ac^n$, $-d^2x$, $-4a^mx^n$, and $-c^2$.
Ans., $-80a^{m+1}c^{m+n+2}d^3x^{n+1}$.

6. Multiply $3x^{\frac{1}{2}}$ by $2x^{\frac{1}{4}}$. *Prod.*, $6x^{\frac{3}{4}}$.

7. Multiply $60a^{\frac{2}{3}}$ by $8a^{\frac{1}{2}}$. *Prod.*, $480a^{\frac{7}{6}}$.

8. Multiply $3a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $-7a^{\frac{1}{4}}b^{\frac{1}{3}}$. *Prod.*, $-21a^{\frac{3}{4}}b^{\frac{2}{3}}$.

9. Multiply $5a^{\frac{1}{3}}b^2$ by $7a^{\frac{1}{3}}b^{-2}c$. *Prod.*, $35a^{\frac{2}{3}}c$.

10. Multiply $10a^2$ by $3a^{-2}$. *Prod.*, 30.

11. Multiply $3y^{\frac{1}{n}}$ by $6y^2$. *Prod.*, $18y^{\frac{1+2n}{n}}$.

12. Multiply $13x^{\frac{1}{n}}$ by $5x^{\frac{1}{m}}$. *Prod.*, $65x^{\frac{m+n}{mn}}$.

13. Multiply $-50a^{\frac{1}{x}}$ by $-4a^{\frac{5}{x}}$. *Prod.*, $200a^{\frac{6}{x}}$.

14. Multiply $309a^{\frac{1}{m}}b^m$ by $9a^{\frac{2}{m}}b^n$. *Prod.*, $2781a^{\frac{3}{m}}b^{m+n}$.

15. Multiply $-5x^{-1}y^{-2}$ by $-3x^2y^3$. *Prod.*, $15xy$.

16. Multiply $x^{\frac{2}{3}}$ by $x^{-\frac{1}{2}}$. *Prod.*, $x^{\frac{1}{6}}$.

17. Multiply x^m by x^{-m}. *Prod.*, 1.

18. Multiply a^2b^{-c} by a^3b^c. *Prod.*, a^5.

19. Multiply $\sqrt{ax}$ by a^2x. *Prod.*, $a^{\frac{5}{2}}x^{\frac{3}{2}}$.

20. Multiply $\sqrt[3]{a^2}$ by $\sqrt{a^3}$. *Prod.*, $a^{\frac{13}{6}}$.

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92. **Prob.—To multiply two factors together when one or both are polynomials.**

Rule.—*Multiply each term of the multiplicand by each term of the multiplier, and add the products.*

Demonstration.—Thus, if any quantity is to be multiplied by $a+b-c$, if we take it a times (*i. e.* multiply by a), then b times, and add the results, we have taken it $a+b$ times. But this is taking it c too many times, as the multiplier required it to be taken $a+b$ *minus* c times. Hence we must multiply by c, and subtract this product from the sum of the other two. Now to subtract this product is simply to add it with its signs changed (**77**). But, giving the $-$ sign of c its effect as we multiply, will change the signs of the product, and we can add the partial products as they stand. Q. E. D.

EXAMPLES.

Ex. 1. Multiply $2a-3b+4c$ by $3a+2b-5c$.

Model Solution.—Writing the multiplier under the multiplicand, as a matter of convenience, I have

$$\begin{array}{llllll}
2a & -3b & +4c & & & \\
3a & +2b & -5c & & & \\
\hline
6a^2 & -9ab & +12ac & & & \\
 & +4ab & & -6b^2 & +8bc & \\
 & & -10ac & & +15bc & -20c^2 \\
\hline
6a^2 & -5ab & +2ac & -6b^2 & +23bc & -20c^2
\end{array}$$

Now taking the multiplicand $3a$ times, I have $6a^2-9ab+12ac$. Taking it $2b$ times, I have $4ab-6b^2+8bc$. I have thus taken it too many times, by $5c$ times. Hence I am to take it $5c$ times, and then subtract this partial product from the others. Therefore I multiply by $5c$, and change the signs as I proceed, and finally add the three partial products. I thus obtain $3a+2b-5c$ times the multiplicand.

2. Multiply $x+y$ by $x+y$. *Prod.*, $x^2+2xy+y^2$.

3. Multiply $5x+4y$ by $3x-2y$.
Prod., $15x^2+2xy-8y^2$.

4. Multiply x^2+xy-y^2 by $x-y$.
Prod., $x^3-2xy^2+y^3$.

5. Multiply $2ac^2-3by$ by $2c^3-3y^2$.
Prod., $4ac^5-6bc^3y-6ac^2y^2+9by^3$.

6. Multiply $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.

7. Multiply $a^3+3a^2x+3ax^2+x^3$ by $a^3-3a^2x+3ax^2-x^3$.
Prod., $a^6-3a^4x^2+3a^2x^4-x^6$.

8. Multiply a^2-b^2 by a^2+b^2.

9. Multiply together $(a\overset{\text{I}}{+}b)$, $(a^2+\overset{\text{II}}{ab}+b^2)$, $(a\overset{\text{III}}{-}b)$, and $(a^2-\overset{\text{IV}}{ab}+b^2)$.

Suggestion.—Perform this first by multiplying together I and II, and then III and IV, and taking the product of these products. 2d, By multiplying the product of I and III into the product of II and IV. 3d, By multiplying the product of I and IV by the product of II and III. *Result in each case*, a^6-b^6.

10. Multiply a^3-x^3 by m^2-n^2.
Prod., $a^3m^2-m^2x^3-a^3n^2+n^2x^3$.

11. Multiply $2a^2+2a+5$ by a^2-a.
Prod., $2a^4+3a^2-5a$.

12. Multiply $2x^3+4x^2+8x+16$ by $3x-6$.
Prod., $6x^4-96$

13. Multiply $a+b-c$ by $m-n$.
Prod., $am-an+bm-bn-cm+cn$.

14. Multiply $x^4+x^2y^2+y^4$ by x^2-y^2. *Prod.*, x^6-y^6.

15. Multiply $x^2-4x+16$ by $x+5$.
Prod., $x^3+x^2-4x+80$.

16. Multiply $a^4-a^3y+a^2y^2-ay^3+y^4$ by $a+y$.
Prod., a^5+y^5.

17. Multiply $x^2-50x-100$ by $x+2$.
Prod., $x^3-48x^2-200x-200$.

18. Multiply $2x^2+3x-1$ by $2x^2-3x+1$.
Prod., $4x^4-9x^2+6x-1$.

19. Multiply $x^{\frac{1}{2}}-b^{\frac{2}{3}}$ by x^2+b.
Prod., $x^{\frac{5}{2}}-b^{\frac{2}{3}}x^2+bx^{\frac{1}{2}}-b^{\frac{5}{3}}$.

20. Multiply $a^{\frac{1}{3}}-b^{-\frac{2}{3}}$ by a^2-b.
Prod., $a^{\frac{7}{3}}-a^2b^{-\frac{2}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}}$.

21. Multiply $2a^{-\frac{1}{2}}-3b^{\frac{2}{3}}$ by $2a^{-\frac{1}{2}}+3b^{\frac{2}{3}}$.
Prod., $4a^{-1}-9b^{\frac{4}{3}}$.

22. Multiply $x^{\frac{9}{4}}+x^{\frac{3}{2}}y^{-\frac{3}{4}}+x^{\frac{3}{4}}y^{-\frac{3}{2}}+y^{-\frac{9}{4}}$ by $x^{\frac{3}{4}}-y^{-\frac{3}{4}}$.
Prod., x^3-y^{-3}.

23. Multiply a^m+b^m by a^n+b^n.
Prod., $a^{m+n}+a^nb^m+a^mb^n+b^{m+n}$.

24. Multiply $x^{\frac{1}{3}}+y^{\frac{1}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. *Prod.*, $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

25. Multiply $m^{\frac{2}{3}}+m^{\frac{1}{3}}n^{\frac{1}{3}}+n^{\frac{2}{3}}$ by $m^{\frac{1}{3}}-n^{\frac{1}{3}}$.
Prod., $m-n$.

26. Multiply $3a^{m-1}-2b^{n-2}$ by $2a-3b^2$.
Prod., $6a^m-4ab^{n-2}-9a^{m-1}b^2+6b^n$.

27. Multiply $2a^{m-p}b^{-3p}+a^{-2p}b^p$ by $3a^mb^{3p}-a^{3p}b^{-2p}$.
Prod., $6a^{2m-p}+3a^{m-2p}b^{4p}-2a^{m+2p}b^{-5p}-a^pb^{-p}$.

93. **Definition.**—When an indicated operation is performed the expression is said to be *expanded*.

28. Expand $(a+b)(a-b)$; also $(x+y)(x+y)$; also $(x-a)(x^2+ax+a^2)$; also $(m+n)(m+n)-(m-n)(m-n)$.
Last result, $4mn$.

THREE IMPORTANT THEOREMS.

94. **Theorem.**—*The square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

Demonstration.—Let x be *any* one quantity and y *any other*. The sum is $x+y$; and the square is, the square of the first, x^2, plus twice the product of the two, $2xy$, plus the square of the second, y^2. That is $(x+y)^2=x^2+2xy+y^2$. For $(x+y)^2=(x+y)(x+y)$ which expanded becomes $x^2+2xy+y^2$. Q. E. D.

EXAMPLES.

Ex. 1. What is the square of $2a^2+3x^2$?

Model Solution.—This is the sum of the two quantities $2a^2$ and $3x^2$, and hence by the theorem the square equals the square of the first, $4a^4$, plus twice the product of the two, $12a^2x^2$, plus the square of the second, $9x^4$. $\therefore\ (2a^2+3x^2)^2 = 4a^4+12a^2x^2+9x^4$.

Scholium.—The pupil should give mentally the squares of the following expressions:

2. Square $4a^{\frac{1}{2}}+2x$. *Result*, $16a+16a^{\frac{1}{2}}x+4x^2$.

3. Square $x^{-\frac{2}{3}}+xy$. *Result*, $x^{-\frac{4}{3}}+2x^{\frac{1}{3}}y+x^2y^2$.

4. Square $5a^{-2}+3a^3b^2$. *Result*, $25a^{-4}+30ab^2+9a^6b^4$.

5. Square $\frac{1}{2}ab^{-2}+\frac{2}{3}x^{-1}b^{-1}$.
Result, $\frac{1}{4}a^2b^{-4}+\frac{2}{3}ab^{-3}x^{-1}+\frac{4}{9}x^{-2}b^{-2}$.

---

95. **Theorem.**—*The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Demonstration.—Let x and y be any two quantities. The difference is $x-y$. Now $(x-y)^2 = (x-y)(x-y)$ which expanded gives $x^2-2xy+y^2$. Q. E. D.

EXAMPLES.

Ex. 1. Square $2x-3y$. *Result*, $4x^2-12xy+9y^2$.

2. Square $x^{-1}y-2y^2$. *Result*, $x^{-2}y^2-4x^{-1}y^3+4y^4$.

3. Square $2a^{\frac{m}{n}}-3b^{-\frac{1}{n}}$ *Result*, $4a^{\frac{2m}{n}}-12a^{\frac{m}{n}}b^{-\frac{1}{n}}+9b^{-\frac{2}{n}}$.

4. Square $m^{-p}-n^{-q}$. *Result*, $m^{-2p}-2m^{-p}n^{-q}+n^{-2q}$.

5. Square $3a^{-\frac{1}{m}}-2b^{-\frac{m}{n}}$.
Result, $9a^{-\frac{2}{m}}-12a^{-\frac{1}{m}}b^{-\frac{m}{n}}+4b^{-\frac{2m}{n}}$.

96. Theorem.—*The product of the sum and difference of two quantities is equal to the difference of their squares.*

Demonstration.—Let x and y be any two quantities. Their sum is $x+y$, and their difference is $x-y$. Now $(x+y)$ multiplied by $(x-y)$ gives, by actual mulplication, x^2-y^2, or the difference of the squares of the two quantities. Q. E. D.

EXAMPLES.

Ex. **1.** Find the product of $2a^2+3b$ and $2a^2-3b$.

Model Solution.—Here I have the sum of the two quantities $2a^2$ and $3b$, to be multiplied by their difference. Now $(2a^2+3b)\times(2a^2-3b)$ is, by the theorem, the square of $2a^2$, or $4a^4$, minus the square of the second, or $9b^2$. $\therefore$ $(2a^2+3b)(2a^2-3b)=4a^4-9b^2$

2. Find the product of $a+2b$ and $a-2b$.
Prod., a^2-4b^2.

3. Find the product of $2a+3b$ and $2a-3b$.
Prod., $4a^2-9b^2$.

4. Find the product of $7a+2b$ and $7a-2b$.
Prod., $49a^2-4b^2$.

5. Find the product of $5a^3+6b^2$ by $5a^3-6b^2$.
Prod., $25a^6-36b^4$.

6. Find the product of $m^{\frac{1}{2}}+n^{\frac{1}{2}}$ and $m^{\frac{1}{2}}-n^{\frac{1}{2}}$.
Prod., $m-n$.

7. Find the product of $2^{\frac{1}{2}}x^{\frac{1}{2}}+3^{\frac{1}{2}}y^{\frac{1}{2}}$ and $2^{\frac{1}{2}}x^{\frac{1}{2}}-3^{\frac{1}{2}}y^{\frac{1}{2}}$.
Prod., $2x-3y$.

8. Find the product of $3a^2b^3+2a^{\frac{1}{2}}b^{\frac{3}{4}}$ and $3a^2b^3-2a^{\frac{1}{2}}b^{\frac{3}{4}}$.
Prod., $9a^4b^6-4ab^{\frac{3}{2}}$.

SYNOPSIS FOR REVIEW.

- **MULTIPLICATION.**
  - **DEFINITIONS.**
    - **Multiplication.**
    - **Multiplier.**
    - **Multiplicand.**
    - **Product.**
    - **Factors.**
    - COR.
      - 1. Multiplier, abstract.
      - 2. Product like multiplicand.
      - SCH.—How the sign of a multiplier is to be understood.
  - **FUNDAMENTAL PROPOSITIONS.**
    - **1. Order of Factors.** DEM.
      - 1. Two factors.
      - 2. More than two.
      - SCH.—Multiplicand concrete.
    - **2. Law of Signs.**
      - DEM.
        - $+ \times +$, $- \times -$ } Give +.
        - $+ \times -$, $- \times +$ } Give —.
      - COR.
        - 1. Positive Factors.
        - 2. Even No. Negative Factors.
        - 3. Odd No. Negative Factors.
    - **3. Laws of Exponents.** DEM.
      - 1. Positive Integer.
      - 2. Positive Fraction.
      - 3. Negative Integer.
      - Negative Fraction.
  - **TO PERFORM MULTIPLICATION.**
    - **Prob. 1.** WHAT? RULE. Demonstration.
    - **Prob. 2.** WHAT? RULE. Demonstration.
  - **THREE IMPORTANT THEOREMS.**
    - **1. Square of Sum.** Demonstration.
    - **2. Square of Difference.** Demonstration.
    - **3. Product of Sum and Difference.** Dem.

Test Questions.—State and demonstrate the law of the signs in multiplication. How are expressions consisting of the same quantity affected by exponents, multiplied? How many cases arise? Demonstrate each. What is the difference between the square of the sum, the square of the difference, and the product of the sum and difference of two quantities? Demonstrate. What is the product of a^3 and a^4? Prove it. What of a^5 and a^{-2}? Prove it. What of $a^{\frac{2}{3}}$ and $a^{\frac{2}{5}}$? Prove it. What of a^m and a^{-m}? Prove it.

DIVISION

SECTION V.

97. **Division** is the process of finding how many times one quantity is contained in another.

The **Dividend** is the quantity to be divided. The **Divisor** is the quantity by which we divide. The **Quotient** is the result, which shows how many times the divisor is contained in the dividend. The **Remainder** is what is left of the dividend after the integral part of the quotient is produced.

98. **Scholium 1.**—Division is the converse of multiplication. Since a product consists of (contains) as many times the multiplicand as there are units in the multiplier, the multiplier shows how many times the multiplicand is contained in the product. The product, therefore, corresponds to the dividend, the multiplicand to the divisor, and the multiplier to the quotient. But, as in multiplication, multiplicand and multiplier may change places without affecting the product, either of them may be considered as divisor and the other as quotient, the product being the dividend.

99. **Scholium 2.**—In accordance with the last scholium, the problem of division may be stated: *Given the product of two factors and one of the factors, to find the other; and the sufficient reason for any quotient is, that multiplied by the divisor it gives the dividend.*

100. **Cor. 1.**—*Dividend and divisor may both be multiplied or both be divided by the same number without affecting the quotient.*

101. **Cor. 2.**—*If the dividend be multiplied or divided by any number, while the divisor remains the same, the quotient is multiplied or divided by the same.*

102. **Cor. 3.**—*If the divisor be multiplied by any number while the dividend remains the same, the quotient is divided by that number; but if the divisor be divided, the quotient is multiplied.*

103. **Cor. 4.**—*The sum of the quotients of two or more quantities divided by a common divisor, is the same as the quotient of the sum of the quantities divided by the same divisor.*

104. **Cor. 5.**—*The difference of the quotients of two quantities divided by a common divisor, is the same as the quotient of the difference divided by the same divisor.*

These corollaries are all immediate consequences of the notion of division. They need no *demonstration*, but it is well that they be fully *illustrated*. Thus Cor. 1, may be illustrated as follows:

If a given number of apples are divided among any number of boys, each boy will receive just the same number as if twice or thrice as many were divided among twice or thrice as many boys, or as if $\frac{1}{2}$ or $\frac{1}{3}$ as many were divided among $\frac{1}{2}$ or $\frac{1}{3}$ as many boys.

Cor. 4 may be illustrated thus: As 2 is contained in 8 *four* times and in 6 *three* times, it is contained in $8+6$ *four*$+$*three*, or 7 times.

105. **Cancellation** is the striking out of a factor common to both dividend and divisor, and does not affect the quotient, as appears from (**100**).

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106. **Lemma 1.**—*When the dividend is positive the quotient has the same sign as the divisor; but when the dividend is negative, the quotient has an opposite sign from the divisor.*

Demonstration.—This proposition is a direct consequence of the law of the signs in multiplication, since the dividend corresponds to the product, and a positive product arises from like signs in the factors, and a negative product, from unlike signs.

Scholium.—In both multiplication and division, Like *signs give* $+$, *and* Unlike *signs* $-$ *results.*

***107.* Lemma 2.**—*When the dividend and divisor consist of the same quantity affected by exponents, the quotient is the common quantity with an exponent equal to the exponent in the dividend, minus that in the divisor.* That is, $a^m \div a^n = a^{m-n}$, whether m and n be integral or fractional, positive or negative.

Demonstration.—This is an immediate consequence of the law of exponents in multiplication, since, in the corresponding case the exponent of the product was found to be the *sum* of the exponents of the factors. Now, as the dividend is the product of the divisor and quotient, it follows that the exponent of the quotient is the exponent of the dividend minus that of the divisor. Q. E. D.

EXAMPLES.

Ex. 1. Divide a^7 by a^3.

Model Solution. $a^7 \div a^3 = a^4$; since $a^4 \times a^3 = a^7$, and division is finding a factor which multiplied into the divisor produces the dividend (**99**).

2. Divide $m^{\frac{3}{4}}$ by $m^{\frac{1}{3}}$. *Quot.*, $m^{\frac{5}{12}}$.

3. Divide $n^{\frac{m}{n}}$ by n^{-1}. *Quot.*, $n^{\frac{m}{n}+1}$, or $n^{\frac{m+n}{n}}$.

4. Divide $(ab)^{2m}$ by $(ab)^{\frac{1}{n}}$. *Quot.*, $(ab)^{2m-\frac{1}{n}}$, or $(ab)^{\frac{2mn-1}{n}}$.

5. Divide a^3 by a^5. *Quot.*, a^{-2}, or $\frac{1}{a^2}$.

6. Divide a^{-2} by a^3. *Quot.*, a^{-5}, or $\frac{1}{a^5}$.

7. Divide $x^{-\frac{1}{3}}$ by x^{-2}. *Quot.*, $x^{\frac{5}{3}}$.

8. Divide x^{-2} by $x^{-\frac{1}{3}}$. *Quot.*, $x^{-\frac{5}{3}}$, or $\frac{1}{x^{\frac{5}{3}}}$.

***108.* Cor. 1.**—*Any quantity with an exponent* 0 *is* 1, *since it may be considered as arising from dividing a quantity by itself.*

Thus x^0 may be considered as $x^m \div x^m$ which is 1, because dividend and divisor are equal. But by the law of exponents $x^m \div x^m$

$= x^0$. $\therefore x^0 = 1$. In like manner the significance of any quantity with an exponent 0 may be explained. $5^0 = 1$, for $\frac{5^2}{5^2} = 1$, and also $\frac{5^2}{5^2} = 5^0$. $\therefore 5^0 = 1$.

109. Cor. 2.—*Negative exponents arise from division when there are more factors of any number in the divisor than in the dividend.*

Thus $a^3 \div a^5 = a^{3-5}$ (**107**) $= a^{-2}$. Again, $a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}$. $\therefore a^{-2} = \frac{1}{a^2}$, which accords with the definition of negative exponents (**43**).

110. Cor. 3.—*A factor may be transferred from dividend to divisor (or from numerator to denominator of a fraction, which is the same thing), and vice versa, by changing the sign of its exponent.*

Thus $\frac{a^{-2}}{b^3} = \frac{1}{a^2b^3}$; for $\frac{a^{-2}}{b^3} = \frac{\frac{1}{a^2}}{b^3} = \frac{1}{a^2} \div b^3 = \frac{1}{a^2b^3}$. Thus also $\frac{a^{-2}}{b^3} = a^{-2}b^{-3}$, since $\frac{a^{-2}}{b^3} = a^{-2} \times \frac{1}{b^3}$, and $\frac{1}{b^3} = b^{-3}$.

EXAMPLES.

Ex. **1.** Show that $\frac{a^{-2}b^2}{x^{-2}y^3} = \frac{b^2x^2}{a^2y^3}$.

Model Solution.—Since $a^{-2} = \frac{1}{a^2}$ and $x^{-2} = \frac{1}{x^2}$, I have $\frac{a^{-2}b^2}{x^{-2}y^3}$

$$= \frac{\frac{1}{a^2}b^2}{\frac{1}{x^2}y^3} = \frac{\frac{b^2}{a^2}}{\frac{y^3}{x^2}} = \frac{b^2}{a^2} \div \frac{y^3}{x^2} = \frac{b^2}{a^2} \times \frac{x^2}{y^3} = \frac{b^2x^2}{a^2y^3}.$$

2. In like manner show that $\frac{2a^{-\frac{1}{2}}x^{-\frac{2}{3}}y}{3m^2n^{-1}x^2} = \frac{2ny}{3a^{\frac{1}{2}}m^2x^{\frac{8}{3}}}$.

3. Free $\frac{5cd^{-1}bx^{-4}}{8a^{-2}x^2y^{-1}z}$ from negative exponents and explain the process.

111. Prob. 1.—To divide one monomial by another.

Rule—*Divide the numerical coefficient of the dividend by that of the divisor, and to the quotient annex the literal factors, affecting each with an exponent equal to its exponent in the dividend minus that in the divisor, and suppressing all factors whose exponents thus become 0.*

The sign of the quotient will be + when dividend and divisor have like signs, and − when they have unlike signs.

Demonstration.—The dividend being the product of divisor and quotient, contains all the factors of both; hence the quotient consists of all the factors which are found in the dividend and not in the divisor. Or, the correctness of the rule appears from the fact that it is the converse of the corresponding operation in multiplication, so that quotient and divisor multiplied together produce the dividend. The law for the sign of the quotient is demonstrated in (**106**). Q. E. D.

EXAMPLES.

Ex. 1. Divide $12a^3x^4b$ by $3a^2x^2$.

Model Solution.

Operation. $3a^2x^2\,\overline{)\,12a^3x^4b}$

$4ax^2b$

Explanation.—In the divisor there is a factor 3, hence there must be a factor 4 in the quotient, to produce 12 in the dividend. So, also, since there are 3 factors of a in the dividend, and 2 in the divisor, there must be 1 in the quotient in order that the product of divisor and quotient may be the dividend. In like manner 4 factors of x in the dividend, and 2 in the divisor, require 2 in the quotient. There being 1 factor b in the dividend, and *none* in the divisor, one factor of b must appear in the quotient. Hence $12a^3x^4b \div 3a^2x^2 = 4ax^2b$, which quotient containing all the factors of the dividend not found in the divisor, will, when multiplied into the divisor, produce the dividend.

2. Divide $12b^7c^3y^3$ by $4bc^2y^3$. *Quot.*, $3b^6c$.

3. Divide $8cdx$ by dx. *Quot.*, $8c$.

4. Divide $10a^5cx^2y$ by $5a^4x^2$. *Quot.*, $2acy$.

5. Divide $18a^3b^2x$ by $-3a^2bx$. *Quot.*, $-6ab$.

6. Divide $-20x^3y^4$ by $-5x^2y^2$. *Quot.*, $4xy^2$.

7. Divide $-42a^3m^2y^3$ by $7a^2m^2$. *Quot.*, $-6ay^3$.

8. Divide $-81a^5bx^2y$ by $9a^4by$. *Quot.*, $-9ax^2$.

9. Divide $-13x^2y^2$ by $-13xy^2$. *Quot.*, x.

10. Divide $8a^2c^3$ by $-8a^2c^3$. *Quot.*, -1.

11. Divide $3a^2m$ by $3a^2m$. *Quot.*, 1.

12. Divide $-x^2y^3$ by $x^{-1}y^{-2}$. *Quot.*, $-x^3y^5$.

13. Divide x^m by x^n. *Quot.*, x^{m-n}.

14. Divide $5a^{-2}b^4$ by $2ab$. *Quot.*, $\frac{5b^3}{2a^3}$.

15. Divide $6a^{\frac{1}{3}}b^{\frac{1}{2}}$ by $3a^{\frac{1}{2}}b^{\frac{1}{3}}$. *Quot.*, $2a^{-\frac{1}{6}}b^{\frac{1}{6}}$.

16. Divide $17a^{-\frac{2}{3}}b^{-\frac{1}{2}}x^2$ by $11a^{-2}b^{-1}x^3$.
Quot., $\frac{17}{11}a^{\frac{4}{3}}b^{\frac{1}{2}}x^{-1}$.

17. Divide $a^pb^qc^{-s}$ by ab^2c^3. *Quot.*, $a^{p-1}b^{q-2}c^{-(s+3)}$.

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112. Prob. 2.—To divide a polynomial by a monomial.

Rule.—*Divide each term of the polynomial dividend by the monomial divisor, and write the results in connection with their own signs.*

Demonstration.—This rule is simply an application of the corollaries (**103, 104**); since to divide a polynomial is to find the quotient of the sum or difference of several quantities, which by these corollaries is shown to be the sum or difference of the quotients of the parts. Q. E. D.

EXAMPLES.

Ex. 1. Divide $15a^3x^2+24a^2x^3b-12a^5xc$ by $3ax^3$.

Model Solution.

Operation. $$3ax^3\,\big)\,\underline{15a^3x^2 + 24a^2x^3b - 12a^5xc}$$
$$5a^2x^{-1} + 8ab - 4a^4x^{-2}c$$

Explanation.—I write the divisor on the left of the dividend, and the quotient underneath as a convenient form. Considering the first two terms, the quotient of their sum is the sum of their quotients (**103**); hence I divide each separately and add the results, obtaining $5a^2x^{-1}+8ab$. Again, the quotient of the difference between the sum of these two terms and the third is the difference of the quotients (**104**); hence I subtract from the quotient of the first two the quotient arising from dividing $12a^5xc$, which is $4a^4x^{-2}c$, and have for the entire quotient $5a^2x^{-1}+8ab-4a^4x^{-2}c$.

2. Divide $3a^3b^2-18a^4b^2+6a^2b$ by $3ab$.
Quot., a^2b-6a^3b+2a.

3. Divide $24x^2y^2-8x^4y^5-24xy^2$ by $8x$.
Quot., $3xy^2-x^3y^5-3y^2$.

4. Divide $21a^3x^3-7a^2x^2+14ax$ by $7ax$.
Quot., $3a^2x^2-ax+2$.

5. Divide $42a^3-11a^2+28a$ by $7a$.
Quot., $6a^2-\frac{11}{7}a+4$.

6. Divide $9k^{14}-24k^3c+48k^5$ by $3k^4$.
Quot., $3k^{10}-8k^{-1}c+16k$.

7. Divide $72a^5c^2-48a^7c^4-32a^9c^5$ by $16a^3c^3$.
Quot., $\frac{9}{2}a^2c^{-1}-3a^4c-2a^6c^2$.

8. Divide $36m^{\frac{7}{2}}-48m^{\frac{3}{4}}$ by $4m^{\frac{1}{3}}$.
Quot., $9m^{\frac{19}{6}}-12m^{\frac{5}{12}}$.

9. Divide $m^{\frac{3}{4}}-m^{\frac{3}{5}}n^{\frac{2}{3}}$ by $m^{\frac{1}{4}}$. *Quot.*, $m^{\frac{1}{2}}-m^{\frac{7}{20}}n^{\frac{2}{3}}$.

10. Divide $a^{\frac{5}{6}}-a^{\frac{1}{6}}b^2+a^{\frac{4}{6}}$ by $a^{\frac{1}{6}}$. *Quot.*, $a^{\frac{2}{3}}-b^2+a^{\frac{1}{2}}$.

11. Divide $11a^{\frac{5}{9}}-33a^{\frac{4}{3}}$ by $11a^{\frac{2}{9}}$. *Quot.*, $a^{\frac{1}{3}}-3a^{\frac{10}{9}}$.

12. Divide $72m^{\frac{10}{12}}-60m^{\frac{1}{6}}n^{\frac{2}{3}}$ by $24m^{\frac{1}{6}}$.

13. Divide $2a^4+4a^3b+2a^2b^2$ by $2a^2$.

14. Divide $11a^2x^3-3ab+18x^2$ by $3ax$.

Quot., $\frac{11}{3}ax^2-\frac{b}{x}+6\frac{x}{a}$.

15. Divide $a^{m+1}-a^{m+2}-a^{m+3}-a^{m+4}$ by a^3.

Quot., $a^{m-2}-a^{m-1}-a^{m}-a^{m+1}$.

16. Divide $5x^{\frac{m}{n}}-10x^{-m}+15x^{\frac{1}{n}}y$ by $5x^2$.

Quot., $x^{\frac{m-2n}{n}}-2x^{-(m+2)}+3x^{\frac{1-2n}{n}}y$.

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113. **Definition.**—A polynomial is said to be arranged with reference to a certain letter when the term containing the highest exponent of that letter is placed first at the left or right, the term containing the next highest exponent next, etc.

Illustration.—The polynomial $6x^2y^2+4xy^3+4x^3y+y^4+x^4$, when arranged according to the *descending* powers of y, becomes $y^4+4xy^3+6x^2y^2+4x^3y+x^4$. In this form it also *chances* to be arranged with reference to the ascending powers of x.

114. **Prob. 3.—To perform division when both dividend and divisor are polynomials.**

Rule.—*I. Arrange dividend and divisor with reference to the same letter.*

II. Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

III. Subtract from the dividend the product of the divisor into this term of the quotient, and bring down as many terms to the remainder as may be necessary to form a new dividend. Divide as before, and continue the process till the work is complete.

The demonstration of this rule will be more readily comprehended after the solution of an example.

Ex.—Divide $6a^2x^2+x^4-4ax^3+a^4-4a^3x$ by x^2+a^2-2ax.

Model Solution.

$$\begin{array}{lrrrrrl}
\text{DIVISOR.} & & & \text{DIVIDEND.} & & & \text{QUOTIENT.}\\
a^2-2ax+x^2\,) & a^4 & -4a^3x & +6a^2x^2 & -4ax^3 & +x^4 & (\,a^2-2ax+x^2\\
& a^4 & -2a^3x & +a^2x^2 & & & \\ \cline{2-4}
& & -2a^3x & +5a^2x^2 & -4ax^3 & & \\
& & -2a^3x & +4a^2x^2 & -2ax^3 & & \\ \cline{3-5}
& & & a^2x^2 & -2ax^3 & +x^4 & \\
& & & a^2x^2 & -2ax^3 & +x^4 & \\ \cline{4-6}
\end{array}$$

Explanation.—Having arranged the dividend and divisor with reference to the descending powers of a, and placed the divisor on the left of the dividend, I divide a^4, the first term of the arranged dividend, by a^2, the first term in the arranged divisor, and get a^2 as the highest power of a in the quotient. Now, as I want to find how many times $a^2-2ax+x^2$ is contained in the dividend, and have found it contained a^2 times (and more), I can take this a^2 times the divisor out of the dividend, and then proceed to find how many times the divisor is contained in what is left of the dividend. Hence I multiply the divisor by a^2 and subtract it from the dividend, leaving $-2a^3x+5a^2x^2-4ax^3+x^4$. The same course of reasoning can be applied to this part. Thus I know that the next highest power of a in the quotient will result from dividing the first term of this remainder by the first term of the divisor, etc. When this process has terminated I have taken a^2, and $-2ax$, and x^2 times the divisor out of the dividend, and finding nothing remaining, I know that the dividend contains the divisor just $a^2-2ax+x^2$ times.

We will now give the demonstration of the rule.

Demonstration.—The arrangement of dividend and divisor according to the same letter enables us to find the term in the quotient containing the highest (or lowest if we put the lowest power of the letter first in our arrangement) power of the same letter, and so on for each succeeding term.

The other steps of the process are founded on the principle, that the product of the divisor into the several parts of the quotient is equal to the dividend. Now by the operation, the product of the divisor into the *first* term of the quotient is subtracted from the dividend; then the product of the divisor into the *second* term of the quotient; and so on, till the product of the divisor into each term of the quotient, that is, the product of the divisor into the *whole* quotient, is taken from the dividend. If there is no remainder, it is evident that this product is *equal* to the dividend. If there *is* a remainder, the product of the divisor and quotient is equal to the whole of the dividend *except* the remainder. And this remainder is not included in the parts subtracted from the dividend, by operating according to the rule.

EXAMPLES.

1. Divide $x^3-3ax^2+3a^2x-a^3$ by $x-a$.
Quot., $x^2-2ax+a^2$.

2. Divide $2y^3-19y^2+26y-16$ by $y-8$.
Quot., $2y^2-3y+2$.

3. Divide $x^4-a^2x^2+2a^3x-a^4$ by x^2-ax+a^2.
Quot., x^2+ax-a^2.

4. Divide a^4+4b^4 by $a^2-2ab+2b^2$.

Operation.

$$\begin{array}{lllll}
a^2-2ab+2b^2\,) & a^4+4b^4 & & & (\,a^2+2ab+2b^2 \\
 & \underline{a^4-2a^3b+2a^2b^2} & & & \\
 & & 2a^3b-2a^2b^2+4b^4 & & \\
 & & \underline{2a^3b-4a^2b^2+4ab^3} & & \\
 & & & 2a^2b^2-4ab^3+4b^4 & \\
 & & & \underline{2a^2b^2-4ab^3+4b^4} &
\end{array}$$

5. Divide $8a^2-26ab+15b^2$ by $4a-3b$.
Quot., $2a-5b$.

6. Divide a^3-b^3 by $a-b$. *Quot.*, a^2+ab+b^2.

7. Divide $a^4-4a^3x+6a^2x^2-4ax^3+x^4$ by $a^2-2ax+x^2$.
Quot., $a^2-2ax+x^2$.

8. Divide a^3+x^3 by $a+x$. *Quot.*, a^2-ax+x^2.

9. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
Quot., $24x^2-2ax-35a^2$.

10. Divide $2a^2+a-6$ by $2a-3$. *Quot.*, $a+2$.

11. Divide $x^2+7x+10$ by $x+2$. *Quot.*, $x+5$.

12. Divide $x^3-5x^2-46x-40$ by $x+4$.
Quot., $x^2-9x-10$.

13. Divide $x^3-9x^2+27x-27$ by $x-3$.
Quot., x^2-6x+9.

[**Note.**—In the following the pupil will need to observe whether the terms are properly arranged or not before commencing the division.]

14. Divide $5x^2y+y^3+x^3+5xy^2$ by $4xy+y^2+x^2$.
Quot., $x+y$.

15. Divide $6x^2y^2-4xy^3-4x^3y+y^4+x^4$ by $x-y$.
Quot., $x^3-3x^2y+3xy^2-y^3$.

16. Divide $6x^4-96$ by $3x-6$.
Quot., $2x^3+4x^2+8x+16$.

17. Divide $3a^2b^4-3a^4b^2-b^6+a^6$ by $a^3-b^3+3ab^2-3a^2b$.
Quot., $a^3+3a^2b+3ab^2+b^3$.

18. Divide $x^5-y^5-5x^4y+5xy^4+10x^3y^2-10x^2y^3$ by $x-y$.
Quot., $(x-y)^4$.

19. Divide a^5-b^5 by $a-b$.
Quot., $a^4+a^3b+a^2b^2+ab^3+b^4$.

20. Divide m^4+n^4 by $m+n$.

21. Divide $32x^5+243$ by $2x+3$.

22. Divide b^4-3y^4 by $b-y$.
Quot., $b^3+b^2y+by^2+y^3-\dfrac{2y^4}{b-y}$.

23. Divide x^2+px+q by $x+a$.

Quot., $x+p-a+\dfrac{q-pa+a^2}{x+a}$.

24. Divide a^4+b^4 by $a^2+ab\sqrt{2}+b^2$.

Quot., $a^2-ab\sqrt{2}+b^2$.

25. Divide $\frac{1}{2}x^3+x^2+\frac{3}{8}x+\frac{3}{4}$ by $\frac{1}{2}x+1$. *Quot.*, $x^2+\frac{3}{4}$.

Operation.

$$
\begin{array}{r@{\,}l}
\tfrac{1}{2}x+1\,) & \tfrac{1}{2}x^3+x^2+\tfrac{3}{8}x+\tfrac{3}{4}\,(\underline{x^2+\tfrac{3}{4}} \\
 & \underline{\tfrac{1}{2}x^3+x^2\qquad\qquad\quad} \\
 & \qquad\qquad\;\;\tfrac{3}{8}x+\tfrac{3}{4} \\
 & \qquad\qquad\;\;\underline{\tfrac{3}{8}x+\tfrac{3}{4}}
\end{array}
$$

26. Divide x^4+y^4 by $x+y$.

Quot., $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$.

27. Divide $a^{2m}+2a^mb^n+b^{2n}$ by a^m+b^n.

Quot., a^m+b^n.

Operation.

$$
\begin{array}{r@{\,}l}
a^m+b^n\,) & a^{2m}+2a^mb^n+b^{2n}\,(\underline{a^m+b^n} \\
 & \underline{a^{2m}+\;a^mb^n\qquad\quad} \\
 & \qquad\quad a^mb^n+b^{2n} \\
 & \qquad\quad \underline{a^mb^n+b^{2n}}
\end{array}
$$

28. Divide $2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}$ by a^n-b^n.

Quot., $2a^{2n}-4a^nb^n+2b^{2n}$.

29. Divide $\frac{1}{3}x^3+\frac{17}{6}x^2-\frac{5}{4}x+\frac{9}{4}$ by $\frac{1}{3}x+3$.

Quot., $x^2-\frac{1}{2}x+\frac{3}{4}$.

30. Divide $x^{\frac{3}{2}}-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. *Quot.*, $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

31. Divide $a-b$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$.

Quot., $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$.

32. Divide $x^{\frac{5}{2}}-x^2-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}}$ by $x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2$.

Quot., $x-x^{\frac{1}{2}}$.

33. Divide $a^{2m}-3a^mc^n+2c^{2n}$ by a^m-c^n.

Quot., a^m-2c^n.

34. Divide $x^3+5x+\frac{5}{x}+\frac{1}{x^3}$ by $x+\frac{1}{x}$.

Quot., $x^2+4+\frac{1}{x^2}$.

35. Divide $x^4-\frac{1}{x^4}$ by $x-\frac{1}{x}$. *Quot.*, $x^3+x+\frac{1}{x}+\frac{1}{x^3}$.

36. Divide $\frac{a^3}{x}-2x^2-\frac{3a^2}{2}+\frac{3x^3}{a}$ by $2a-3x$.

Quot., $\frac{a^2}{2x}-\frac{x^2}{a}$.

37. Divide $\frac{x^3}{y^3}+\frac{5x^2}{12y^2}+\frac{39}{16}$ by $\frac{x}{3y^2}+\frac{1}{2y}$.

Quot., $\frac{3x^2}{y}-\frac{13x}{4}+\frac{39y}{8}$.

38. Divide $a^2+(a-1)x^2+(a-1)x^3+(a-1)x^4-x^5$ by $a-x$.

Operation.

$$
\begin{array}{l}
a-x\,)\ a^2+(a-1)x^2+(a-1)x^3+(a-1)x^4-x^5\ (\underline{a+x+x^2+x^3+x^4} \\
\qquad \underline{a^2 \qquad - ax \qquad\qquad} \\
\qquad\qquad\qquad ax\ +(a-1)x^2 \\
\qquad\qquad\qquad \underline{ax\ - \qquad\quad x^2 \qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad ax^2+(a-1)x^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{ax^2- \qquad\quad x^3 \qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad ax^3+(a-1)x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{ax^3- \qquad\quad x^4 \qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ax^4-x^5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{ax^4-x^5}
\end{array}
$$

39. Divide $x(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$. *Quot.*, $(x-1)a+x^2$.

Scholium.—This process of division is strictly analogous to "Long Division" in common arithmetic. The arrangement of the terms corresponds to the regular order of succession of the thousands, hundreds, tens, units, etc., while the other processes are precisely the same in both.

SYNOPSIS FOR REVIEW.

- DIVISION.
  - DEFINITIONS AND DIRECT CONSEQUENCES.
    - **Division.**
    - **Dividend.**
    - **Divisor.**
    - **Quotient.**
    - **Remainder.**
      - Sch. 1.—Relation of division and multiplication.
      - Sch. 2.—General reason for a quotient.
      - Cor. 1.—Multiplying or dividing dividend and divisor.
      - Cor. 2.—Multiplying or dividing dividend.
      - Cor. 3.—Multiplying or dividing divisor.
      - Cor. 4.—Quotient of sum.
      - Cor. 5.—Quotient of difference.
      - Definition.—Cancellation.
  - LEMMAS AND DIRECT CONSEQUENCES.
    - **Laws of Signs.**
      - DEMONSTRATION.
    - **Laws of Exponents.**
      - DEM.
        - Cor. 1.—Meaning of exponent 0.
        - Cor. 2.—Negative exponents.
        - Cor. 3.—Transferring exponents.
  - PROBLEMS.
    - **1. To divide one monomial by another.** RULE. DEM.
    - **2. To divide a polynomial by a monomial.** RULE. DEM.
    - **3. To divide one polynomial by another.** RULE. DEM.
    - **Scholium.**

Test Questions.—How do negative exponents arise, and what do they signify? What is the value of any quantity with 0 for its exponent? Why? How do you divide, when dividend and divisor consist of the same quantity affected by exponents? Why? Give the General Rule (**Prob. 3**) and its demonstration.

Factoring

CHAPTER II.

SECTION I.

FUNDAMENTAL PROPOSITIONS.

115. **The Factors** of a number are those numbers which multiplied together produce it.

A *Factor* is, therefore, a *Divisor*. A *Factor* is also frequently called a *measure*, a term arising in Geometry.

116. **A Common Divisor** is a common integral factor of two or more numbers.

The Greatest Common Divisor of two or more numbers is the greatest common integral factor, or the product of all the common integral factors.

Common Measure and *Common Divisor* are equivalent terms.

117. **A Common Multiple** of two or more numbers is an integral number which contains each of them as a factor, or which is divisible by each of them.

The Least Common Multiple of two or more numbers is the least integral number which is divisible by each of them.

118. **A Composite Number** is one which is composed of integral factors different from itself and unity.

119. **A Prime Number** is one which has no integral factor other than itself and unity.

120. Numbers are said to be *Prime to each other* when they have no common integral factor other than unity.

Scholium.—The above definitions and distinctions have come into use from considering Decimal Numbers. They are only applicable to literal numbers in an accommodated sense. Thus, in the general view which the literal notation requires, all numbers are composite in the sense that they can be factored; but as to whether the factors are greater or less than unity, integral or fractional, we cannot affirm.

121. **Prop. 1.**—*A monomial may be resolved into literal factors by separating its letters into any number of groups, so that the sum of all the exponents of each letter shall make the exponent of that letter in the given monomial.*

Illustration. $5a^2bx^{\frac{8}{5}}$ may be resolved into $5a \cdot ab^{\frac{1}{2}} \cdot ab^{\frac{1}{2}} \cdot x^{\frac{8}{5}}$, or $5a^2 \times b^{\frac{1}{3}} \times a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{5}} \times a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{5}} \times x^{\frac{1}{5}}$, or into any number of factors, in a like manner.

Demonstration.—This is a direct result of the principle that monomials are multiplied by writing the several letters in connection, and affecting each with an exponent equal to the sum of the exponents of that letter in the factors.

EXAMPLES.

Ex. **1.** Separate $12a^2bx^3$ into all the possible factors with positive integral exponents.

Factors, 3, 2, 2, a, a, b, x, x, and x.

2. Separate $16a^{\frac{3}{5}}x^{\frac{2}{3}}$ into two equal factors.

Factors, $4a^{\frac{3}{10}}x^{\frac{1}{3}}$, and $4a^{\frac{3}{10}}x^{\frac{1}{3}}$.

3. Separate $8x^{\frac{2}{5}}y^3$ into two factors, one of which is $4xy$.

Factors, $4xy$, and $2x^{-\frac{3}{5}}y^2$.

4. Remove the factor $2(ax)^{\frac{1}{2}}$ from $6a^2x$. *Result*, $3a^{\frac{3}{2}}x^{\frac{1}{2}}$.

5. Remove the factor $3a^{\frac{1}{2}}$ from $15ac^2$. *Result*, $5\sqrt{a}c^2$.

6. Resolve m into two factors.

Result, $m^{\frac{1}{2}} \cdot m^{\frac{1}{2}}$, $\sqrt{m} \cdot \sqrt{m}$, $m^{\frac{2}{3}} \cdot m^{\frac{1}{3}}$, $m^{\frac{3}{4}} \cdot m^{\frac{1}{4}}$, $m^{\frac{3}{5}} \cdot m^{\frac{2}{5}}$, etc.

7. Resolve x into two equal factors. Into 3. Into 5.

---

122. Prop. 2.—*Any factor which occurs in every term of a polynomial can be removed by dividing each term of the polynomial by it.*

Demonstration.—This is the ordinary problem of division by a monomial, since divisor and quotient are the factors of the dividend.

EXAMPLES.

1. Factor $3a-3b$. *Result*, $3(a-b)$.
2. Factor $ax-bx+cx$. *Result*, $(a-b+c)x$.
3. Factor $5-5y$. *Result*, $5(1-y)$.
4. Factor $6a^2y^3-18ay^2$. *Result*, $6ay^2(ay-3)$.
5. Factor $42x^2y-14xy+7x^3y^2$.
   Result, $7xy(6x-2+x^2y)$.
6. Factor $81cm^2x-63cmx^3$. *Result*, $9cmx(9m-7x^2)$.
7. Factor $7a^3x^2y-21ax^2y^3$.
   Factors, $7ax^2y$ and a^2-3y^2.
8. Factor $72ab^2x^3-84ab^3x^2+96a^2b^2x^2$.
   Factors, $12ab^2x^2$ and $6x-7b+8a$.
9. Factor $924a^4b^3c^3-1078a^3b^4c^3+1232a^3b^3c^4$.
   Factors, $14a^3b^3c^3$ and $66a-77b+88c$.

---

123. Prop. 3.—*If two terms of a trinomial are* POSITIVE *and the third term is twice the product of the square roots of these two, and* POSITIVE, *the trinomial is the square of the* SUM *of these square roots. If the third term is* NEGATIVE, *the trinomial is the square of the* DIFFERENCE *of the two roots.*

Demonstration.—This is a direct consequence of the theorems that, "The square of the sum of two quantities is the sum of their squares *plus* twice their product;" and "The square of their difference is the sum of their squares *minus* twice their product." **(94, 95.)**

EXAMPLES.

Ex. 1. What are the factors of $a^2+2ab+b^2$?

Model Solution.—I observe that the two terms a^2 and b^2 of this trinomial are both positive, and that the other term, $2ab$, is twice the product of the square roots of a^2, and b^2, and positive. Hence $a^2+2ab+b^2=(a+b)^2=(a+b)(a+b)$. Therefore, the factors are $a+b$ and $a+b$.

2. What are the factors of $x^2-2ax+a^2$?

3. What are the factors of $m^4+n^4+2m^2n^2$?

Suggestion.—Here m^4 and n^4 are both positive, and $2m^2n^2$ is the product of their square roots. *Ans.*, (m^2+n^2) and (m^2+n^2).

4. What are the factors of $16a^2-8a+1$?

5. What are the factors of $m+2\sqrt{mn}+n$?

Ans., $(\sqrt{m}+\sqrt{n})$ and $(\sqrt{m}+\sqrt{n})$.

6. What are the factors of $x^{\frac{2}{3}}+y^{\frac{4}{5}}-2x^{\frac{1}{3}}y^{\frac{2}{5}}$?

Ans., $x^{\frac{1}{3}}-y^{\frac{2}{5}}$ and $x^{\frac{1}{3}}-y^{\frac{2}{5}}$.

7. What are the factors of $a^2b+ab^2+2a^{\frac{3}{2}}b^{\frac{3}{2}}$?

8. What are the factors of $x^2+2xy-y^2$?

Query.—Can the last be factored according to this *Proposition*? Why?

9. If $4a^2$, $16b^2$, and $16ab$ are the terms of a trinomial, what must be their respective signs so that the trinomial can be factored?

10. Factor $16a^6b^4m^2-8a^3b^2m+1$.

Factors, $(4a^3b^2m-1)$ and $(4a^3b^2m-1)$.

11. Factor $9a^2+3ab+\frac{1}{4}b^2$.

Factors, $(3a+\frac{1}{2}b)$ and $(3a+\frac{1}{2}b)$.

12. Factor $49a^2b^2-\frac{28}{3}ab^3+\frac{4}{9}b^4$.

Factors, $(7ab-\frac{2}{3}b^2)$ and $(7ab-\frac{2}{3}b^2)$

13. Factor $\frac{a^2}{b^2}+2+\frac{b^2}{a^2}$. *Factors*, $\left(\frac{a}{b}+\frac{b}{a}\right)$ and $\left(\frac{a}{b}+\frac{b}{a}\right)$.

14. Factor $\frac{4}{25}x^6+\frac{1}{16}y^8-\frac{1}{5}x^3y^4$.
Factors, $(\frac{2}{5}x^3-\frac{1}{4}y^4)$ and $(\frac{2}{5}x^3-\frac{1}{4}y^4)$.

15. Factor $\frac{x^4}{y^4}+\frac{y^4}{x^4}-2$.
Factors, $\left(\frac{x^2}{y^2}-\frac{y^2}{x^2}\right)$ and $\left(\frac{x^2}{y^2}-\frac{y^2}{x^2}\right)$.

16. Factor $a^2+2a\sqrt{x}+x$.
Factors, $(a+\sqrt{x})$ and $(a+\sqrt{x})$.

17. Factor $x-2b\sqrt{x}+b^2$.
Factors, $(\sqrt{x}-b)$ and $(\sqrt{x}-b)$.

18. Factor $x-2\sqrt{x\cdot y}+y$.
Factors, $(\sqrt{x}-\sqrt{y})$ and $(\sqrt{x}-\sqrt{y})$.

19. Factor $a^2b-2ax\sqrt{b}+x^2$.
Factors, $(a\sqrt{b}-x)$ and $(a\sqrt{b}-x)$.

20. Factor $a+2\sqrt{a}+1$.
Factors, $(\sqrt{a}+1)$ and $(\sqrt{a}+1)$.

***124.* Prop. 4.**—*The difference between two quantities is equal to the product of the sum and difference of their square roots.*

Demonstration.—This is an immediate consequence of the theorem, that "The product of the sum and difference of two quantities is the difference of their squares." Thus $a^2-b^2=(a+b)(a-b)$. Also, if we have $m-n$, m is the square of $m^{\frac{1}{2}}$, and n, of $n^{\frac{1}{2}}$; $\therefore m-n=(m^{\frac{1}{2}}+n^{\frac{1}{2}})(m^{\frac{1}{2}}-n^{\frac{1}{2}})$, etc.

EXAMPLES.

Ex. 1. Factor $16x^2-9y^2$.
Factors, $(4x-3y)$ and $(4x+3y)$.

2. Factor $a^2y^2-b^2x^2$.

Factors, $(ay-bx)$ and $(ay+bx)$.

3. Factor $16a^2x^2-25b^2y^2$.

Factors, $(4ax-5by)$ and $(4ax+5by)$.

4. Factor x^4-y^4. *Factors,* (x^2-y^2) and (x^2+y^2).

5. Factor x^6-y^6; also x^8-y^8; also $x^{\frac{2}{3}}-y^{\frac{2}{3}}$; also $x^{-4}-y^{-4}$; also $4a^{-6}-9b^{-4}$.

Factors of last three, $(x^{\frac{1}{3}}-y^{\frac{1}{3}})$ *and* $(x^{\frac{1}{3}}+y^{\frac{1}{3}})$; $(x^{-2}-y^{-2})$ *and* $(x^{-2}+y^{-2})$; *and* $(2a^{-3}-3b^{-2})$ *and* $(2a^{-3}+3b^{-2})$.

6. Factor $m-n$.

Factors, $(\sqrt{m}-\sqrt{n})$ and $(\sqrt{m}+\sqrt{n})$.

7. Factor $2a-4b^2$.

Factors, $2, (\sqrt{a}-\sqrt{2}b)$ and $(\sqrt{a}+\sqrt{2}b)$.

8. Factor $25x^m-3y^{2n}$.

Factors, $(5x^{\frac{m}{2}}-\sqrt{3}y^n)$ and $(5x^{\frac{m}{2}}+\sqrt{3}y^n)$.

125. Prop. 5.—*When one of the factors of a quantity is given, to find the other, divide the given quantity by the given factor, and the quotient will be the other.*

Demonstration.—This is the ordinary problem of division, since the divisor and quotient are the factors of the dividend. Any problem in division affords an example.

EXAMPLES.

Ex. 1. Resolve $2a^2-36a^3b$ into two factors one of which is $2a^2$.

2. Remove the factor $\frac{1}{x^2}-\frac{3}{y^3}$ from $\frac{1}{x^4}-\frac{9}{y^6}$.

3. Remove the factor $\frac{a^2}{b}+\frac{3c}{2m}$ from $a^4b^{-2}-\frac{9c^2m^{-2}}{4}$.

4. Remove the factor $3a^{-2}+2x^{-1}$ from $\frac{9}{a^4}+\frac{12}{a^2x}+\frac{4}{x^2}$.

126. Prop. 6.—*The difference between any two quantities is a divisor of the* DIFFERENCE *between the same powers of the quantities.*

The SUM *of two quantities is a divisor of the* DIFFERENCE *of the same* EVEN *powers, and the* SUM *of the same* ODD *powers of the quantities.*

Demonstration.—Let x and y be any two quantities and n any positive integer. *First*, $x-y$ divides x^n-y^n. *Second*, if n is *even*, $x+y$ divides x^n-y^n. *Third*, if n is *odd*, $x+y$ divides x^n+y^n.

FIRST.

$$\begin{array}{l}
x-y\,)\,x^n-y^n \qquad (\underline{x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3+\text{etc.}} \\
\qquad\quad \underline{x^n-x^{n-1}y} \\
\qquad\qquad x^{n-1}y-y^n \\
\qquad\qquad \underline{x^{n-1}y-x^{n-2}y^2} \\
\qquad\qquad\qquad\quad x^{n-2}y^2-y^n \\
\qquad\qquad\qquad\quad \underline{x^{n-2}y^2-x^{n-3}y^3} \\
\qquad\qquad\qquad\qquad\qquad x^{n-3}y^3-y^n \\
\qquad\qquad\qquad\qquad\qquad \underline{x^{n-3}y^3-x^{n-4}y^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x^{n-4}y^4-y^n
\end{array}$$

Taking the first case, we proceed in form with the division, till enough terms to determine the law are found. We find that each remainder consists of two terms, the second of which, $-y^n$, is the second term of the dividend constantly brought down unchanged; and the first contains x with an exponent decreasing by unity in each successive remainder, and y with an exponent *increasing* at the same rate that the exponent of x *decreases*. At this rate the exponent of x in the nth remainder becomes 0, and that of y, n. Hence the nth remainder is y^n-y^n or 0; and the division is exact.

SECOND AND THIRD.

$$\begin{array}{l}
x+y\,)\,x^n\pm y^n \qquad (x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3,\ \text{etc.} \\
\qquad\quad \underline{x^n+x^{n-1}y} \\
\qquad\qquad -x^{n-1}y\pm y^n \\
\qquad\qquad \underline{-x^{n-1}y-x^{n-2}y^2} \\
\qquad\qquad\qquad\quad x^{n-2}y^2\pm y^n \\
\qquad\qquad\qquad\quad \underline{x^{n-2}y^2+x^{n-3}y^3} \\
\qquad\qquad\qquad\qquad\qquad -x^{n-3}y^3\pm y^n \\
\qquad\qquad\qquad\qquad\qquad \underline{-x^{n-3}y^3-x^{n-4}y^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x^{n-4}y^4\pm y^n
\end{array}$$

Taking $x+y$ for a divisor, we observe that the exponent of x in the successive remainders decreases, and that of y increases the same as before. But now we observe that the first term of the remainder is $-$ in the *odd* remainders, as the 1st, 3rd, 5th, etc., and $+$ in the even ones, as the 2nd, 4th, 6th, etc. Hence if n is *even, and the second term of the dividend is* $-y^n$, the nth remainder is $y^n - y^n$ or 0, and the division is exact. Again, if n is *odd*, and the second term of the dividend is $+y^n$, the nth remainder is $-y^n + y^n$ or 0, and the division is exact. Q. E. D.

127. Scholium.—The pupil should notice carefully the form of the quotient in each of the above cases, and be able to write it without dividing. He should also be able to tell at a glance, whether such a division is exact or not, and if not exact, what the remainder, or fractional term of the quotient is.

EXAMPLES.

Write the quotients in the following without dividing:

1. $(x^5 - y^5) \div (x - y)$.
2. $(a^3 + b^3) \div (a + b)$. *Quot.*, $a^2 - ab + b^2$.
3. $(a^3 - b^3) \div (a - b)$.
4. $(a^4 - m^4) \div (a + m)$.
5. $(a^4 - m^4) \div (a - m)$.
6. $(m^7 + n^7) \div (m + n)$.
7. $(m^8 - b^8) \div (m + b)$, also by $(m - b)$.
8. $(x^{12} - y^{12}) \div (x^3 + y^3)$.

Suggestion.—Notice that x^{12} is the 4th (an *even*) power of x^3, and y^{12} is the *same* power of y^3. It may be best for the pupil to obtain this quotient thus. Put $x^3 = a$, and $y^3 = b$. Then $x^{12} = a^4$ and $y^{12} = b^4$. But $(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3$. Whence restoring the values of a and b, we have $(x^{12} - y^{12}) \div (x^3 + y^3) = x^9 - x^6y^3 + x^3y^6 - y^9$, since $a^3 = x^9$, $-a^2b = -x^6y^3$, $+ab^2 = +x^3y^6$, and $-b^3 = -y^9$.

9. $(m^{10} + n^{10}) \div (m^2 + n^2)$.

   Quot., $m^8 - m^6n^2 + m^4n^4 - m^2n^6 + n^8$.
10. $(x^{18} - y^{18}) \div (x^3 - y^3)$, also by $(x^3 + y^3)$.
11. $(a^3m^6 - y^3) \div (am^2 - y)$.

Suggestion.—Notice that a^3m^6 is the third power of am^2, as y^3 is of y. The quotient may be obtained by first using a single letter for am^2, as x, writing out $(x^3-y^3)\div(x-y)=x^2+xy+y^2$, and finally restoring the value of x. After a little practice the pupil will not need to go through such a process, but can write the quotient at once. This quotient is $a^2m^4+am^2y+y^2$.

12. $(1-y^5)\div(1-y)$.

13. $(y^6-1)\div(y+1)$, also by $(y-1)$.

14. $(a^5x^5+b^{10}y^{10})\div(ax+b^2y^2)$.

15. $(16x^4-81y^8)\div(2x+3y^2)$, also by $(2x-3y^2)$.

[**Note.**—In the following examples if the division is not exact, the quotient should be written with the fractional term.]

16. Is a^7+b^7 exactly divisible by $a+b$, or $a-b$?

17. Is m^6-y^6 exactly divisible by $m+y$, or $m-y$?

18. Write the following quotients $(x^{2m}-y^{2m})\div(x+y)$, also by $x-y$. $(x^{2m+1}+y^{2m+1})\div(x+y)$, m being an integer.

Query.—Is $2m$ even or odd? Is $2m+1$ even or odd?

FOR REVIEW OR ADVANCED COURSE.

[**Note.**—The following corollary and examples may be omitted till after the pupil has been through the chapter on Powers and Roots, if thought desirable.]

128. Cor.—*Proposition 6 applies equally to cases involving fractional or negative exponents.*

Thus, $x^{\frac{1}{5}}-y^{\frac{1}{5}}$ divides $x^{\frac{4}{5}}-y^{\frac{4}{5}}$, since the latter is the difference between the 4th powers of $x^{\frac{1}{5}}$ and $y^{\frac{1}{5}}$. So in general $x^{-\frac{n}{m}}-y^{-\frac{s}{r}}$ divides $x^{-\frac{an}{m}}-y^{-\frac{as}{r}}$, a being any positive integer. This becomes evident by putting $x^{-\frac{n}{m}}=v$, and $y^{-\frac{s}{r}}=w$; whence $x^{-\frac{an}{m}}=v^a$, and $y^{-\frac{as}{r}}=w^a$. But v^a-w^a is divisible by $v-w$, hence $x^{-\frac{an}{m}}-y^{-\frac{as}{r}}$ is divisible by $x^{-\frac{n}{m}}-y^{-\frac{s}{r}}$.

EXAMPLES.

19. What is the quotient of $(a^{-5}-b^{-5})\div(a^{-1}-b^{-1})$?

Ans., $a^{-4}+a^{-3}b^{-1}+a^{-2}b^{-2}+a^{-1}b^{-3}+b^{-4}$.

20. What is the quotient of $(x^{-\frac{4}{5}}-y^{-\frac{4}{5}})\div(x^{-\frac{1}{5}}+y^{-\frac{1}{5}})$?

Ans., $x^{-\frac{3}{5}}-x^{-\frac{2}{5}}y^{-\frac{1}{5}}+x^{-\frac{1}{5}}y^{-\frac{2}{5}}-y^{-\frac{3}{5}}$.

21. What is the quotient of $\left[\frac{a^4}{x^4}-y^8\right]\div\left[\frac{a}{x}+y^2\right]$?

22. Is $x^{-\frac{7}{5}}-y^{-7}$ divisible by $x^{-\frac{1}{5}}-y^{-1}$, or $x^{-\frac{1}{5}}+y^{-1}$?

23. Is x^3-y^3 divisible by $\sqrt{x}+\sqrt{y}$, or $\sqrt{x}-\sqrt{y}$?

24. Is x^3+y^3 divisible by $\sqrt{x}+\sqrt{y}$, or $\sqrt{x}-\sqrt{y}$?

25. What is the quotient of $(ax^{\frac{5}{2}}+b^{\frac{5}{3}}y)\div(a^{\frac{1}{5}}\sqrt{x}+\sqrt[3]{b}y^{\frac{1}{5}})$?

***129*. Prop. 7.**—*A trinomial can be resolved into two binomial factors, when one of its terms is the product of the square root of one of the other two, into the sum of the factors of the remaining term. The two factors are severally the algebraic sum of this square root, and each of the factors of the third term.*

Illustration.—Thus, in $x^2+7x+10$, we notice that $7x$ is the product of the square root of x^2, and $2+5$ (the sum of the factors of 10). Therefore, the factors of $x^2+7x+10$ are $x+2$ and $x+5$. Again $x^2-3x-10$, has for its factors $x+2$ and $x-5$, $-3x$ being the product of the square root of x^2 (or x), and the sum of -5 and 2 (or -3), which are factors of -10. Still again, $x^2+3x-10=(x-2)(x+5)$, determined in the same manner.

Demonstration.—The truth of this proposition appears from considering the product of $x+a$ by $x+b$, which is $x^2+(a+b)x+ab$. In this product, considered as a trinomial, we notice that the term $(a+b)x$, is the product of $\sqrt{x^2}$ and $a+b$, the sum of the factors of

ab. In like manner $(x+a)(x-b)=x^2+(a-b)x-ab$, and $(x-a)(x-b)=x^2-(a+b)x+ab$, both of which results correspond to the enunciation. Q. E. D.

[**Note.**—In application, Proposition 7 requires the solution of the problem: Given the sum and product of two numbers to find the numbers, the complete solution of which cannot be given at this stage of the pupil's progress. It will be best for him to rely, at present, simply upon inspection.]

EXAMPLES.

Ex. **1.** Show that, according to this proposition, the factors of x^2+3x+2 are $x+1$ and $x+2$. Verify it by actual multiplication.

2. In like manner show the following:

$$x^2-7x+12=(x-3)(x-4).$$
$$x^2-x-12=(x+3)(x-4).$$
$$x^2+x-12=(x-3)(x+4).$$

3. Factor x^2-5x+6, $x^2+3x-28$, $x^2-11x+28$, $x^2+2x-35$, x^2-2x-8, x^2+2x-8, $72-y-y^2$, $5-4z-z^2$.

***130.* Prop. 8.**—*We can often detect a factor by separating a polynomial into parts.*

Demonstration.—The correctness of this process depends upon the principle that whatever divides the parts, divides the whole.

EXAMPLES.

Ex. **1.** Factor $x^2+12x-28$.

Model Solution.—The form of this polynomial, suggests that there may be a binomial factor in it, *or in a part of it.* Now x^2-4x+4 is the square of $x-2$, and $(x^2-4x+4)+(16x-32)$ makes $x^2+12x-28$. But $(x^2-4x+4)+(16x-32)=(x-2)(x-2)+(x-2)16=(x-2)(x-2+16)=(x-2)(x+14)$. Whence $x-2$, and $x+14$ are seen to be the factors of $x^2+12x-28$.

2. Resolve $12a^2b+3by^2-15aby$ into its factors.

Solution. $3b$ is seen to be a common factor, and removing it, we have $4a^2+y^2-5ay$. This latter polynomial may be separated into the parts $a^2-2ay+y^2$, and $3a^2-3ay$. Hence, $4a^2+y^2-5ay = (a^2-2ay+y^2)+(3a^2-3ay) = (a-y)(a-y)+(a-y)3a = (a-y)(a-y+3a) = (a-y)(4a-y)$. Hence the factors of $12a^2b+3by^2-15aby$, are $3b$, $a-y$, and $4a-y$.

3. Factor $6a^5+15a^4b-4a^3c^2-10a^2bc^2$.
Factors, a^2, $3a^2-2c^2$, and $2a+5b$.

4. Factor $3a^2-2a-1$. *Factors,* $a-1$ and $3a+1$.

5. Factor $5a^2-8ax+3x^2$. *Factors,* $5a-3x$ and $a-x$.

6. Factor $20ax-25an-16bx+20bn$.

7. Factor $8a^2+22ab+15b^2$.

8. Factor $12x^3-8xy-9x^2y^2+6y^3$.

9. Factor $42x^2-2xy-20y^2$, $42x^2-58xy+20y^2$, and $42x^2+58xy+20y^2$.

10. Factor $64xy-10ax+96by-15ab$.

11. Factor $a^2nx-a^2mn-a^2xy+a^2my$.

12. Factor $45a^3x^3+30a^2bx^2y-27a^2cx^2y-18abcxy^2$.

13. Factor $12x^5-8x^3y^2+21x^2y-14y^3$.

COMMON DIVISORS

SECTION II.

131. **The Highest Divisor** of a literal number is the divisor of the highest degree with reference to the letter of arrangement, *i. e.*, the divisor which involves the highest exponent of that letter.

The Highest Common Divisor of two or more literal numbers is the divisor of highest degree which is common to all of them.

Scholium.—While, with reference to decimal numbers, the term *greatest divisor* is appropriate, it is scarcely proper to apply it to literal quantities, for the values of the letters not being fixed, or specific, *great* or *small* cannot be affirmed of them. Thus, whether a^3 is greater than a, depends upon whether a is greater or less than 1, to say nothing of its character as positive or negative. So, also, we cannot with propriety call a^3-y^3 *greater* than $a-y$. If $a=\frac{1}{2}$, and $y=\frac{1}{4}$, $a^3-y^3=\frac{7}{64}$, and $a-y=\frac{1}{4}$; $\therefore$ in this case $a^3-y^3<a-y$. Again, if a and y are both greater than 1, but $a<y$, a^3-y^3 though *numerically* greater than $a-y$ is *absolutely* less, since *it is a greater negative.*

[**Note.**—The general rule for finding the Greatest or Highest Common Divisor is founded upon the four following lemmas.]

132. Lemma 1.—*The Greatest or Highest C. D. of two or more numbers is the product of their common prime factors.*

Demonstration.—Since a factor and a divisor are the same thing, all the common factors are all the common divisors. And, since the product of any number of factors of a number is a divisor of that number, the product of *all* the *common prime factors* of two or more numbers is a *common divisor* of those numbers. Moreover, this product is the *Greatest* or *Highest* C. D., since no other factor can be introduced into it without preventing its measuring (dividing) at least one of the given numbers. Q. E. D.

EXAMPLES.

Ex. 1. What is the G. C. D. of 48, 108, and 72?

Model Solution. $48=2\cdot2\cdot2\cdot2\cdot3$, $108=2\cdot2\cdot3\cdot3\cdot3$, and $72=2\cdot2\cdot2\cdot3\cdot3$. The common factors, or divisors, are $2\cdot2\cdot3$; hence, $2\cdot2\cdot3=12$, divides all the numbers, and is a C. D. Moreover, as there is no other common factor, if we introduce an additional factor into $2\cdot2\cdot3$, it will not divide the given number or numbers which do not contain this factor. Thus, if we introduce a factor 5, it will not divide any one of the given numbers. If we introduce another factor, 2, and have $2\cdot2\cdot3\cdot2=24$, it will not divide 108, which has but two factors of 2. $\therefore$ $2\cdot2\cdot3=12$, is the G. C. D. of 48, 108, and 72.

[Solve the following, giving the explanation as above.]

2. Find the G. C. D. of 84, 126, and 210. *G. C. D.*, 42.

3. Find the G. C. D. of 70, 105, and 245. *G. C. D.*, 35.

4. Find the Highest C. D. of $12ab^2c$ and $25b^4c^5$.

Solution.—Here we see at once the *Highest* Common Divisor with reference to the letters, as b, b, and c are all the common literal factors. There is no common factor in 12 and 25. Hence b^2c is the H. C. D., inasmuch as no other factor can be introduced into this product (b^2c) and it still remain a divisor of $12ab^2c$ and $25b^4c^5$.

5. Find the H. C. D. of $12a^3b^5c^2$ and $8a^2b^3c$.
H. C. D., $4a^2b^3c$.

6. Find the H. C. D. of $6a^5x^2$, $18a^3xy$, and $24a^2x^3y^2$.

7. Find the H. C. D. of $3x^2y^{\frac{2}{3}}$ and $2mnxy$.
H. C. D., $xy^{\frac{2}{3}}$.

8. Find the H. C. D. of $3a^2x-6abx+3b^2x$ and $4a^2y-4b^2y$.

Suggestions. $3a^2x-6abx+3b^2x=3x(a-b)(a-b)$; and $4a^2y-4b^2y=4y(a+b)(a-b)$. $\therefore$ H. C. D. is $a-b$.

9. Find the H. C. D. of $x^2+12x-28$ and $x^3+9x^2+27x-98$.

Suggestions.—Factor the polynomial of the lower order first, as it is more easily resolved. $x^2+12x-28$, may be factored by (**129**) or (**130**). According to the latter process we have $x^2+12x-28=x^2-4x+4+(16x-32)=(x-2)^2+16(x-2)=(x-2)(x-2+16)=(x-2)(x+14)$. Now find by trial whether either or both of these factors are divisors of the other polynomial. $x-2$ is, and $x+14$ is not. $\therefore$ $x-2$ is the H. C. D.

10. Find the H. C. D. of $3a^2x^2y-3a^2xy-36a^2y$ and $3a^2x^3-48a^2x-3a^2x^2+48a^2$. *H. C. D.*, $3a^2x-12a^2$.

133. **Scholium.**—The difficulty of factoring renders this process impracticable in many cases. There is a more general method. But, in order to demonstrate the rule, we must prove three additional lemmas.

134. **Lemma 2.**—*A polynomial of the form* $Ax^n+Bx^{n-1}+Cx^{n-2}$ - - - - $Ex+F$ *which has no common factor in every term, has no divisor of its own degree except itself.*

By this form it is meant that all like powers of the letter of arrangement are collected into a single term, and that the polynomial is arranged according to some letter, as for division.

Demonstration.—1st. Such a polynomial cannot have one factor of the nth degree,—its own,—with reference to the letter of arrangement, and another which contains the letter of arrangement, for the product of two *such* factors would be of a higher (or different) degree from the given polynomial.

2d. It cannot have a factor of the nth degree with reference to the letter of arrangement, and another factor which does not contain that letter, for this last factor would appear as a common factor in every term, which is contrary to the hypothesis. Q. E. D.

Illustration. $4a^2-5ay+y^2$ cannot have a factor containing a^2, such as a^2-y, and another which contains a, as $4a-2y$, since two *such* factors multiplied together, would give a higher power of a than a^2, to say nothing of other terms. Again, it cannot have such a factor as a^2-ay-y and another which does not contain a, as $4-5y+y^2$, since this last would then appear as a factor in every term of the product when arranged with reference to a; as $(4-5y+y^2)a^2-(4-5y+y^2)ay-(4-5y+y^2)y$. But the hypothesis is that the given polynomial shall have no common factor in every term.

135. Lemma 3.—*A divisor of any number is a divisor of any multiple of that number.*

Illustration.—This is an axiom. If a is contained in b, q times, it is evident that it is contained in n times b, or nb, n times q, or nq times.

136. Lemma 4.—*A common divisor of two numbers is a divisor of their sum and also of their difference.*

Demonstration.—Let a be a C. D. of m and n, contained in m, p times, and in n, q times. Then $(m+n) \div a = p + q$. Q. E. D.

Illustration.—If this appears a little abstract and unsatisfactory to the learner, let him illustrate it thus: 4 is contained in 20, 5 times, and in 8, 2 times; hence it is contained in $20+8$, $5+2$, or 7 times, and in $20-8$, $5-2$, or 3 times.

137. **Prob.—To find the H. C. D. of two polynomials without the necessity of resolving them into their prime factors.**

Rule.—*I. Arranging the polynomials with reference to the same letter, and uniting into single terms the like powers of that letter, remove any common factor or factors which may appear in all the terms of both polynomials, reserving them as factors of the H. C. D.*

II. Reject from each polynomial all factors which appear in it and not in the other.

III. Taking the polynomials, thus reduced, divide the one with the greatest exponent of the letter of arrangement, by the other, continuing the division till the exponent of the letter of arrangement is less in the remainder than in the divisor.

IV. Reject any factor which occurs in every term of this remainder, and divide the divisor by the remainder as thus reduced, treating the remainder and last divisor as the former polynomials were. Continue this process of rejecting factors from each term of the remainder, and dividing the last divisor by the last remainder till nothing remains.

If, at any time, a fraction would occur in the quotient, multiply the dividend by any number that will avoid the fraction.

The last divisor multiplied by all the first reserved common factors of the given polynomials, will be the H. C. D. sought.

Demonstration.—We will first give a demonstration of this rule by means of a particular example. Let it be required to find the H. C. D. with reference to a, of the following polynomials:

$12a^2b^2+3b^2y^2-15ab^2y+12a^2by+3by^3-15aby^2$, and $6a^3b^2-6a^2b^2y-2b^2y^3+2ab^2y^2+6a^3by-6a^2by^2-2by^4+2aby^3$.

Operation.

$$\begin{array}{lr}
12a^2b^2 + 3b^2y^2 - 15ab^2y + 12a^2by + 3by^3 - 15aby^2 & (A) \\
\underline{6a^3b^2 - 6a^2b^2y - 2b^2y^3 + 2ab^2y^2 + 6a^3by - 6a^2by^2 - 2by^4 + 2aby^3} & (B) \\
4a^2b + by^2 - 5aby + 4a^2y + y^3 - 5ay^2 & (C) \\
\underline{3a^3b - 3a^2by - by^3 + aby^2 + 3a^3y - 3a^2y^2 - y^4 + ay^3} & (D) \\
(4b + 4y)\,a^2 - (5by + 5y^2)\,a + (by^2 + y^3) & (E) \\
\underline{(3b + 3y)\,a^3 - (3by + 3y^2)\,a^2 + (by^2 + y^3)\,a - (by^3 + y^4)} & (F)
\end{array}$$

$$\begin{array}{lll}
 & (G) & (H) \\
 & 4a^2-5ya+y^2\,)\;3a^3-3ya^2+y^2a-y^3 & \\
 & \underline{4} & \\
(I) & 12a^3-12ya^2+4y^2a-4y^3\,(\,3a & \\
(K) & \underline{12a^3-15ya^2+3y^2a} & \\
(L) & 3ya^2+y^2a-4y^3 & \\
 & \underline{4} & \\
(M) & 12ya^2+4y^2a-16y^3\,(\,3y & \\
(N) & \underline{12ya^2-15y^2a+3y^3} & \\
(O) & \text{Reject } 19y^2. \qquad \underline{19y^2a-19y^3} & (G) \\
(P) & a-y\,)\;4a^2-5ya+y^2\,(\,4a-y & \\
 & \underline{4a^2-4ya} & \\
 & -ya+y^2 & \\
 & \underline{-ya+y^2} &
\end{array}$$

∴ The H. C. D. of (A) and (B) is $(b)(b+y)(a-y) = ab^2+aby-b^2y-by^2$.

Reasoning.

1st. Removing b from both (A) and (B), and reserving it as a factor of the H. C. D. (LEM. 1), and rejecting 3 from (A), and 2 from (B), since they are not common factors, and hence cannot enter into the H. C. D., we get (C) and (D).

2nd. Arranging (C) and (D) with reference to a, the letter with respect to which the H. C. D. is sought, both for convenience in dividing, and to observe if any other common factor appears, we have (E) and (F). In these we readily discover and remove the common factor $(b+y)$, reserving it as a factor of the H. C. D., and get (G) and (H).

3rd. Having now found two of the common factors of (A) and (B), and removed some which were not common, it remains to determine whether there are any more common factors, that is, whether (G) and (H) have a C. D.

(G) is its own H. D., Lem. 2; hence if it divides (H), it is the H. C. D. of (G) and (H). We, therefore, try it. But, to avoid fractions, we multiply (H) by 4, since, as 4 is not a factor of (G), the H. C. D. of (G) and 4 times (H) is the same as of (G) and (H). We thus obtain (I). ∴ If (G) divides (I) it is the H. C. D. of (G) and (H). Trying it, we find the remainder (L). Now, any C. D. of (G) and (I) is the divisor of (K) [a multiple of (G)], Lem. 3, and of (L), Lem. 4.

We now have to find the H. C. D. of (G) and (L), upon which we reason just as upon (G) and (H). Thus, as (G) is its own *only* divisor of the 2nd degree, if it divides (L), or (M) [since (M) = 4 (L) and 4 is not a factor in (G)], it is the H. C. D. of (G) and (H). Trying, we find a remainder (O). Now, any divisor of (G) and (M) is a divisor of (N), Lem. 3, and of (O), Lem. 4. The question is therefore reduced to finding the H. C. D. of (G) and (O). Upon these we reason as before.

Rejecting $19y^2$ since it is not a factor of (G), and hence, cannot enter into the H. C. D., we have (P). The H. C. D. of (G) and (P) cannot be higher than (P), and as (P) is its own only divisor of its own degree, if it divides (G,) it is the H. C. D. sought. It does. ∴ $a-y$ is the H. C. D. of (G) and (H).

Finally, b, $(b+y)$, and $(a-y)$ are all the common factors of (A) and (B), and hence, $b\ (b+y)\ (a-y) = ab^2+aby-b^2y-by^2$ is their H. C. D.

EXAMPLE.

Find the H. C. D. with respect to x, of $x^4-8x^3+21x^2-20x+4$, and $2x^3-12x^2+21x-10$.

Model Solution.—Calling the 2nd, (A) and the 1st (B), I have the following:

$$\begin{array}{lr}
\quad (A) \qquad\qquad\qquad \textbf{Operation.} & (B) \qquad\qquad\qquad\qquad \\
2x^3 - 12x^2 + 21x - 10\ ) & x^4 - 8x^3 + 21x^2 - 20x + 4 \\
 & \overline{} \\
(C) & 2x^4 - 16x^3 + 42x^2 - 40x + 8\ (x - 2 \\
 & \underline{2x^4 - 12x^3 + 21x^2 - 10x\phantom{{}+8}} \\
 & -\ 4x^3 + 21x^2 - 30x + 8 \\
 & \underline{-\ 4x^3 + 24x^2 - 42x + 20} \\
(D) \quad \text{Reject } -3 & \underline{-\ 3x^2 + 12x - 12} \\
(E) & x^2 - 4x + 4
\end{array}$$

$$
\begin{array}{llrl}
 & & (E)\qquad\qquad\quad (A) & \\
 & & x^2-4x+4\,)\,2x^3-12x^2+21x-10\,(\,2x-4 & \\
 & & \underline{2x^3-8x^2+8x} & \\
 & & -4x^2+13x-10 & \\
 & & \underline{-4x^2+16x-16} & \\
(F) & \text{Reject } -3 & \underline{-3x+6} & \qquad (E) \\
(G) & & x-2\,) & x^2-4x+4\,(\,x-2 \\
 & & & \underline{x^2-2x} \\
 & & & -2x+4 \\
 & & & \underline{-2x+4}
\end{array}
$$

Reasoning.

The two given polynomials being arranged with reference to x, and no common, or other factor, appearing, I proceed at once to determine by successive divisions their H. C. D.

The H. C. D. cannot be higher than (A), the lower of the two; and, as it is its own only divisor of the 3rd degree, if it divides (B), it is the H. C. D. But to avoid fractions I multiply (B) by 2, and, since 2 is not a factor of (A), the H. C. D. of (A) and (B), is the H. C. D. of (A) and (C). Now if (A) divides (C) it is the H. C. D. Trying it, I find a remainder, (D).

Again the H. C. D. of (A) and (C) is also a divisor of (D), for (D) is the difference between (C) and ($x—2$) times (A), both of which are divisible by the H. C. D. of (A) and (C). The question is now reduced to finding the H. C. D. of (A) and (D). Upon these I reason exactly as before upon (A) and (B).

Thus, since —3 is a factor of (D), and not of (A), it can be rejected; and the H. C. D. of (A) and (D) is the H. C. D. of (A) and (E). This cannot be higher than (E); and as (E) is its own only divisor of the 2nd degree, if it divides (A), it is the H. C. D. Trying it, I find a remainder, (F). Upon this remainder, and (E), I reason as before, upon (D) and (A).

Thus, the H. C. D. of (E) and (A) is a divisor of (E) and (F), since (F) is the difference between (A) and a multiple ($2x—4$ times) of (E). The question is then reduced to finding the H. C. D. of (E) and (F). Upon these I reason as before, rejecting —3, which is not a common factor, and hence, forms no part of H. C. D. of (E) and (F), and finding by trial that (G) is a divisor of (E). Therefore, $x—2$ is the H. C. D. of (A) and (B).

GENERAL DEMONSTRATION OF THE RULE FOR FINDING THE H. C. D.

Let A and B represent any two polynomials whose H. C. D. is sought.

1st. Arranging A and B with reference to the same letter, for convenience in dividing, and also to render common factors more readily discernible, if any common factors appear, they can be removed and reserved as factors of the H. C. D., since the H. C. D. consists of all the common factors of A and B.

2nd. Having removed these common factors, call the remaining factors C and D. We are now to ascertain what common factors there are in C and D, or to find their H. C. D. As this H. C. D. consists of only the common factors, we can *reject* from each of the polynomials, C and D, any factors which are not common. Having done this, call the remaining factors E and F.

3rd. Suppose polynomial E to be of lower degree with respect to the letter of arrangement than F. (If E and F are of the same degree, it is immaterial which is made the divisor in the subsequent process.) Now, as E is its own only divisor of its own degree (LEM. 2), if it divides F, it is the H. C. D. of the two. If, in attempting to divide F by E to ascertain whether it is a divisor, fractions arise, F can be multiplied by any number not a factor in E (and E has no monomial factor), since the common factors of E and F would not be affected by the operation. Call such a multiple of F, if necessary, F'. Then the H. C. D. of E and F', is the H. C. D. of E and F. If, now, E divides F', it is the H. C. D. of E and F. Trying it, suppose it goes Q times, with a remainder, R.

4th. Any divisor of E and F' is a divisor of R, since $F' - QE = R$, and any divisor of a number divides any multiple of that number (LEM. 3), and a divisor of two numbers divides their difference. The H. C. D. divides E, hence it divides QE, and, as it also divides F', it divides the difference between F' and QE, or R. Therefore, the H. C. D. of E and F', is also the H. C. D. of E. and R.

5th. We now repeat the reasoning of the 3rd and 4th paragraphs concerning E and F, with reference to E and R. Thus, R is by hypothesis of lower degree than E; hence, dividing E by it, rejecting any factor not common to both, or introducing any one into E, which may be necessary to avoid fractions, we ascertain whether R is a divisor of E.

6th. Proceeding thus, till two numbers are found, one of which divides the other, the last divisor is the H. C. D. of E and F, since at every step we show that the H. C. D. is a divisor of the two numbers compared, and the last divisor is its own H. D.

7th. Finally, we have thus found all the common factors of A and B, the product of which is their H. C. D. Q. E. D.

EXAMPLES.

Ex. 1. Find the H. C. D. of $14ax-8a+ax^3-7ax^2$ and $16a^2x^2+6a^2x^4-28a^2x^3$, and give the reasoning as above.
The H. C. D. is $ax-4a$.

2. Find the H. C. D. of $a^3+3a^2b+3ab^2+b^3$ and $5a^5+5b^5$, giving the demonstration. *The H. C. D. is* $a+b$.

3. Find the H. C. D. of $36a^6+9a^3-27a^4-18a^5$ and $27a^5b^2-9a^3b^2-18a^4b^2$. *The H. C. D. is* $9a^4-9a^3$.

#4. Find the H. C. D. of $4xy^2-2y^3+6x^2y$ and $4x^2y+8x^3-4xy^2$. *The H. C. D. is* $2x+2y$.

5. Find the H. C. D. of $3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$. *The H. C. D. is* x^3+3x^2+3x+1.

6. Find the H. C. D. of $2a^3x^4-2a^2bx^3y+2ab^2x^2y^2-2b^3xy^3$ and $4a^2b^2x^3y^2-2ab^3x^2y^3-2b^4xy^4$.
The H. C. D. is $2ax^2-2bxy$.

***138.* Prob.—To find the H. C. D. of three or more polynomials.**

Rule.—*Find the H. C. D. of any two of the given polynomials by one of the foregoing methods, and then find the H. C. D. of this H. C. D. and one of the remaining polynomials. Continue this process till all the polynomials have been used.*

Demonstration.—For brevity, call the several polynomials A, B, C, D, etc. Let the H. C. D. of A and B be represented by P, whence P contains *all* the factors common to A and B. Finding the H. C. D. of P and C, let it be called P′ P′, therefore, contains *all* the common factors of P and C; and as P contains all that are common to A and B, P′ contains all that are common to A, B and C. In like manner if P″ is the H. C. D. of P′ and D, it contains all the common factors of A, B, C, and D, etc. Q. E. D.

EXAMPLES.

Ex. 1. Find the H. C. D. of $2x^4+6x^3+4x^2$, $3x^3+9x^2+9x+6$, and $3x^3+8x^2+5x+2$. *The H. C. D. is* $x+2$.

2. What is the H. C. D. of $10a^5+10a^3b^2+20a^4b$, $2a^3+2b^3$, and $4b^4+12a^2b^2+4a^3b+12ab^3$? *Ans.*, $2(a+b)$.

COMMON MULTIPLES

SECTION III.

139. **The Lowest Multiple** of a given literal number is the number of lowest degree with reference to some specified letter, which contains the given number as a factor.

The Lowest Common Multiple of several numbers is the number of lowest degree which is a multiple of each.

[For the impropriety of the term *Least* Common Multiple as applied to literal numbers, see Art. **131**.]

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140. **Prob.—To find the L. C. M. of two or more numbers.**

Rule.—*I. Take the literal number of the highest degree, or the largest decimal number, and multiply it by all the factors found in the next lower which are not in it.*

II. Again, multiply this product by all the factors found in the next lower number and not in it, and so *continue till all the numbers are used. The product thus obtained is the L. C. M.*

Demonstration.—Let A, B, C, D, etc., represent any numbers arranged in the order of their degrees, or values. Now, as A is its own L. M., the L. C. M. of all the numbers must contain it as a factor. But, in order to contain B, the L. C. M. must contain all the factors of B. Hence, if there are any factors in B which are not found in A, these must be introduced. So, also, if C contains factors not found in A and B, they must be introduced, in order that the product may contain C, etc. Now it is evident that the product so obtained, is the L. C. M. of the several numbers, since it contains all the factors of any one of them, and hence can be divided by any one of them, and if any factor were removed it would cease to be a multiple of some one or more of the numbers. Q. E. D.

Ex. **1.** Find the L. C. M. of (x^3-1), (x^2-1), and $(x+1)$.

Model Solution.—The L. C. M. must contain x^3-1, and as it is its own L. M., if it contains all the factors of the other two, it is the required L. C. M. The factors of x^3-1 are $(x-1)$ and (x^2+x+1). But the product of these does not contain the factors of (x^2-1) which are $(x+1)$ and $(x-1)$. Hence we must introduce the factor $(x+1)$, giving $(x^3-1)(x+1)$, as the L. C. M. of x^3-1 and x^2-1. Now as this product contains the third quantity, it is the L. C. M. of the three.

2. Find the L. C. M. of $(a+b)^2$, a^2-b^2, $(a-b)^2$, and $a^3+3a^2b+3b^2a+b^3$.

Suggestions.—The last is $(a+b)^3$ which contains the factors of the 1st, but neither of the factors of the 3rd. Both factors of $(a-b)^2$ must, therefore, be introduced, giving $(a+b)^3(a-b)^2$ as the L. C. M. of the 1st, 3rd and 4th. And as it contains both factors of a^2-b^2, viz.: $(a+b)$ $(a-b)$, it is the L. C. M. of all.

3. Find the L. C. M. of (x^2-4a^2), $(x+2a)^3$, and $(x-2a)^3$.
L. C. M., $(x^2-4a^2)^3$.

4. Find the L. C. M. of $x^2+2xy+y^2$ and x^3-xy^2.
L. C. M., $(x+y)$ (x^3-xy^2).

5. Find the L. C. M. of $14a^2mx^3$, $35a^3bm^2$, and $7b^4x^2$.
L. C. M., $70a^3b^4m^2x^3$.

6. Find the L. C. M. of $16x^3y^3$, $10a^2x$, and $40a^3x^2$.

L. C. M., $80a^3x^3y^3$.

Scholium.—In applying this rule, if the common factors of the two numbers are not readily discerned, apply the method of finding the H. C. D., in order to discover them.

7. Find the L. C. M. of $x^3-2ax^2+4a^2x-8a^3$, $x^3+2ax^2+4a^2x+8a^3$, and x^2-4a^2.

Model Solution.—The L. C. M. of these numbers must contain $x^3-2ax^2+4a^2x-8a^3$; and as it is its own L. M., if it contains all the factors of $x^3+2ax^2+4a^2x+8a^3$, it is the L. C. M. of these two polynomials. But as the common factors of these numbers, if they have any, are not readily discerned, we apply the method of H. C. D. and find that x^2+4a^2 is the H. C. D. of the two. Since, then, $x^3-2ax^2+4a^2x-8a^3$ contains the factor x^2+4a^2 of the second number, it is only necessary to introduce the other factor in order to have the L. C. M. of the two. Now $(x^3+2ax^2+4a^2x+8a^3)\div(x^2+4a^2)=x+2a$. Hence $(x^3-2ax^2+4a^2x-8a^3)(x+2a)$ or x^4-16a^4 is the L. C. M. of the first two numbers, since it contains all the factors of each, and no more. Now, to find whether x^4-16a^4 is a multiple of the remaining number, x^2-4a^2, or, if it is not, what factors must be introduced to make it so, we proceed in the same way as with the first two numbers. But our first step (or **124**) shows us that x^4-16a^4 *is* a multiple of x^2-4a^2. $\therefore$ x^4-16a^4 is the L. C. M. of the three given numbers.

8. Find the L. C. M. of $6x^2-x-1$ and $2x^2+3x-2$.

L. C. M., $(2x^2+3x-2)(3x+1)$.

9. Find the L. C. M. of $x^3-9x^2+23x-15$ and x^2-8x+7.

L. C. M., $(x^3-9x^2+23x-15)(x-7)=x^4-16x^3+86x^2-176x+105$.

10. Find the L. C. M. of $2x-1$, $4x^2-1$, and $4x^2+1$.

11. Find the L. C. M. of x^3-x, x^3-1, and x^3+1.

L. C. M., $x(x^6-1)=x^7-x$.

12. Find the L. C. M. of $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$, and $x^3-8x^2+19x-12$.

L. C. M., $(x-1)(x-2)(x-3)(x-4)=x^4-10x^3+35x^2-50x+24$.

SYNOPSIS FOR REVIEW.

- **FACTORING.**
  - **DEFINITIONS.**
    - **Factor. Divisor. Measure.**
      - COMMON.
      - G. C. D.
      - H. C. D.
        - DISTINCTION.
    - **Multiple.**
      - COMMON.
      - LEAST C. M.
      - LOWEST C. M.
        - DISTINCTION.
    - **Numbers.**
      - COMPOSITE.
      - PRIME.
      - PRIME TO EACH OTHER.
    - **Scholium.** HOW THESE TERMS ARE APPLIED.
  - **FUNDAMENTAL PROPOSITIONS.**
    - **PROPS.**
      - 1. A MONOMIAL. DEM.
      - 2. A POLYNOMIAL WITH MON. FACTORS. DEM.
      - 3. $a^2 \pm 2ab + b^2$. DEM.
      - 4. $a^2 - b^2$. DEM.
      - 5. ONE FACTOR GIVEN. DEM.
      - 6. $(a^m \pm b^m) \div (a \pm b)$. DEM.
        - *Sch.* Form of quotient.
        - *Cor.* Fractional and negative exponents.
      - 7. A TRINOMIAL. WHEN? HOW? DEM.
      - 8. SEPARATING INTO PARTS.
  - **H. C. D.**
    - **Distinction between G. C. D. and H. C. D.**
    - **Lemmas.**
      - 1. DEM.
      - 2. DEM.
      - 3. DEM.
      - 4. DEM.
    - **General Method.**
      - PROB. 1. RULE. Dem.
      - PROB. 2. RULE. Dem.
  - **L. C. M.**
    - **Distinction between Least and Lowest C. M.**
    - **Problem.** RULE. DEM.
    - **Scholium.** BY MEANS OF H. C. D.

Test Questions.—What are the factors of $a^2 - b^2$? Of $a^2 + 2ab + b^2$? Of $a^2 - 2ab + b^2$? Of $1 - 2x + x^2$? Of $3a^2 + 6a^2x + 3a^2x^2$? Of $x^2 + y^2 + 2xy$? Of $x^2 - x - 12$? Of m? Of a^2b^3y? Of $x^3 - y^3$? State the general rule for testing the divisibility of the sum or difference of like powers. Prove one of the cases, as $(a^m - b^m) \div (a - b)$. Distinction between H. C. D. and G. C. D. Between *Lowest* and *Least* C. M. Explain the process of finding each by factoring. Give the General Rule in each case, and its demonstration.

FRACTIONS

CHAPTER III.

SECTION I.

DEFINITIONS AND FUNDAMENTAL PRINCIPLES.

141. **A Fraction**, in the literal notation, is to be considered as an indicated operation in Division.

A fraction is written, as in common arithmetic, with one number above another and a line between them. The number above the line, *i. e.*, the *dividend*, is called the *Numerator;* and the number below the line, *i. e.*, the divisor, is called the *Denominator*.

Thus: $\frac{2a-5mx^2}{3c+4x^{\frac{1}{2}}}$ means nothing more than $(2a-5mx^2)\div(3c+4x^{\frac{1}{2}})$.

Taken together numerator and denominator are called the *Terms of the Fraction*.

142. **Scholium.**—In the literal notation it becomes impracticable to consider the denominator as indicating the number of equal parts into which unity is divided, and the numerator as indicating the number of those parts represented by the fraction, since the very genius of this notation requires that the letters be not restricted in their signification. Thus in $\frac{a}{b}$, it will not do to say, b represents the number of equal parts into which unity is divided, since the notation requires that whatever conception we take of these quantities should be sufficiently comprehensive to include all values. Hence b may be a mixed number. Now suppose $b=4\frac{2}{5}$. It is absurd to speak of unity as divided into $4\frac{2}{5}$ equal parts.

***143.* The Value of a Fraction** is the quotient of the numerator divided by the denominator.

***144.* Cor. 1.**—*Since numerator is dividend and denominator divisor, we have* (**100, 101, 102**) *the following:*

1. *Dividing or multiplying both terms of a fraction by the same quantity does not alter its value.*

2. *Multiplying or dividing the numerator multiplies or divides the value of the fraction; and*

3. *Multiplying or dividing the denominator divides or multiplies the fraction.*

***145.* Cor. 2.**—*A fraction is multiplied by its denominator by simply removing the denominator.*

This is simply cancelling equal factors from numerator and denominator (Art. **105**). Thus $\frac{3}{4}\times 4=\frac{3\times 4}{4}=3$. In like manner $\frac{a}{x}\times x=\frac{ax}{x}=a$.

146. The terms *Integer* or *Entire Number, Mixed Number, Proper* and *Improper*, are applied to literal numbers, but not with strict propriety. Thus, whether $m+n$ is an integer, a mixed number, or a fraction, depends upon the values of m and n, which the genius of the literal notation requires to be understood as perfectly general, until some restriction is imposed.

For convenience, we adopt the following definitions:

147. A number not having the fractional *form* is said to have the **Integral Form**; as $m+n$, $2c^2d-8a^{-2}x+3x^5y^4$.

148. A polynomial having part of its terms in the fractional and part in the integral form, is called a **Mixed Number**; as $a-x+\frac{cd^2-2y^{\frac{1}{2}}}{am^2}$.

***149.* A Proper Fraction**, in the literal notation, is an expression wholly in the fractional form, and which cannot be expressed in the integral form without negative exponents.

By calling such an expression a proper fraction, we do not assert anything with reference to its value as compared with unity. Thus $\frac{a}{b}$ is a proper fraction, though it may be greater or less than unity. It may also be written ab^{-1}.

***150.* An Improper Fraction** is an expression in the fractional form, but which can be expressed in the integral or mixed form without the use of negative exponents.

Thus, $\frac{2an-8a^2b}{n-3ab}=2a-\frac{2a^2b}{n-3ab}$; the former of which is called an improper fraction.

***151.* A Simple Fraction** is a single fraction with both terms in the integral form.

Thus $\frac{2x-m^2}{4y+2cd}$, and $\frac{2cmx^2}{x-2y}$ are simple fractions.

***152.* A Compound Fraction** is two or more fractions connected by the word *of*; but the expression is not generally applicable in the literal notation.

Thus we may write $\frac{2}{3}$ of $\frac{3}{4}$ with propriety, but not $\frac{a}{b}$ of $\frac{m}{n}$, unless a and b are integral, so that the fraction $\frac{a}{b}$ may be considered as representing equal parts of unity, as $\frac{2}{3}$ does. If the word *of* is considered as simply an equivalent for ×, the notation is, of course, always admissible. But it is scarcely a simple equivalent.

153. **A Complex Fraction** is a fraction having in one or both its terms an expression of the fractional form.

Thus $\dfrac{\frac{1}{2}m-\dfrac{c}{d}}{5\dfrac{x}{y}-a}$, and $\dfrac{\dfrac{ma^2}{cd}}{2b^2x^3}$ are complex fractions.

154. A fraction is in its *Lowest Terms* when there is no common integral factor in both its terms.

155. **The Lowest Common Denominator** is the number of lowest degree, which can form the denominator of several given fractions, giving equivalent fractions of the same values respectively, while the numerators retain the integral form.

156. **Reduction**, in mathematics, is changing the form of an expression without changing its value.

SIGNS OF A FRACTION.

157. In considering the signs of a fraction, we have to notice *three things*, viz.: the sign of the numerator, the sign of the denominator, and the sign before the fraction as a whole. This latter sign does not belong to either the numerator or denominator separately, but to the whole expression. Thus, in the expression $-\dfrac{4a-5cd}{2x+4y^2}$, in the numerator the sign of $4a$ is $+$, and of $5cd$ $-$. In the denominator, the sign of $2x$ is $+$, and of $4y^2$ $+$ also. The sign of the fraction is, $-$. These are the signs of operation. **(56, 33, 34.)**

158. **The essential character** of a fraction, as *positive* or *negative*, can be determined only when the essential character of all the numbers entering into it is known. It may then be determined by principles already given. **(86, 106.)**

EXAMPLES.

Ex. **1.** Is the fraction $-\frac{4a^2-3mx}{2x^3+4y^2}$ essentially positive, or negative, when a, m, x, and y are each negative?

Model Solution.—**1.** The square of $-a$ is $+$, since it is the product of an even number of negative factors (**88**). $\therefore$ $4a^2$ is positive.

For the same reason mx is $+$. $\therefore$ $3mx$ is in itself positive, but it is made negative by the $-$ sign before it.

2. The cube of $-x$ is $-$, since it is the product of an odd number of negative factors (**89**). $\therefore$ $2x^3$ is negative.

The square of $-y$ is $+$ (**88**). $\therefore$ $4y^2$ is positive.

3. Finally, the essential sign of the fraction depends upon the relative values of a, m, x, and y. If $4a^2 > 3mx$, and $2x^3 > 4y^2$, the numerator and denominator have unlike signs, whence the value of the fraction (**113**) would be $-$; but it is made $+$ by the $-$ sign before the fraction.

Similarly, if $4a^2 < 3mx$, and $2x^3 > 4y^2$, the signs of the numerator and denominator are alike, whence the sign of the fraction would be $+$, but the $-$ sign before the fraction would change it to $-$; and the essential character of the fraction would be negative.

2. What is the essential sign of $\frac{2a^2x^2+4y^3}{3b^2-4x}$, when $a=-3$, $x=-2$, $y=-4$, and $b=-5$? *Ans.*, $-$.

3. What is the essential sign of $-\frac{2x^3y+ax^2}{3a^2x}$, when $x=-2$, $a=-1$, and $y=3$? *Ans.*, $-$.

4. What is the essential sign of $-\frac{-3a^2x-4ax^3}{-2y^2-3cx}$, when $x=4$, $a=-2$, $y=-1$, and $c=6$? *Ans.*, $+$.

Reduction

SECTION II.

159. There are five principal reductions required in operating with fractions, viz.; *To Lowest Terms,—From Improper Fractions to Integral or Mixed Forms,—From Integral or Mixed Forms to Improper Fractions,—To Forms having a Common Denominator,*—and from the *Complex to the Simple Form.*

Prob. 1.—To reduce a fraction to its lowest terms.

Rule.—*Reject all common factors from both terms; or divide both terms by their H. C. D.*

Demonstration.—Since the numerator is the dividend and the denominator the divisor, rejecting the same factors from each does not alter the value of the fraction (**100**). Having rejected *all* the common factors, or, what is the same thing, the H. C. D. (which contains all the common factors), the fraction is in its lowest terms (**154**).

EXAMPLES.

Ex. 1. Reduce $\frac{6a^3-6ax^2}{3a^4+6a^3x+3a^2x^2}$ to its lowest terms.

Model Solution.—Resolving the terms of the fraction into their prime factors, I have $\frac{6a^3-6ax^2}{3a^4+6a^3x+3a^2x^2}=\frac{2\cdot\cancel{3}\cdot\cancel{a}(a-x)\cancel{(a+x)}}{\cancel{3}\cdot\cancel{a}\cdot a(a+x)\cancel{(a+x)}}$. Now, cancelling the common factors, 3, a, and $a+x$, which is dividing dividend and divisor by the same quantity and hence does not alter the value of the fraction, I have $\frac{2(a-x)}{a(a+x)}$, or $\frac{2a-2x}{a^2+ax}$. Since in this there is no factor common to numerator and denominator, it is in its lowest terms (**154**).

2. Reduce $\frac{15b^4xy}{25cx^5y^2}$, $\frac{110mx^2y^3}{550m^2xy^5}$, $\frac{12a^2x^3}{36a^3x^5}$, and $\frac{45a^5c^3x^2}{15a^3c^3x}$ to their lowest terms.

2. Reduce $\frac{15b^4xy}{25cx^5y^2}$, $\frac{110mx^2y^3}{550m^2xy^5}$, $\frac{12a^2x^3}{36a^3x^5}$, and $\frac{45a^5c^3x^2}{15a^3c^3x}$ to their lowest terms.

Results (not given in order), $\frac{1}{3ax^2}$, $\frac{x}{5my^2}$, $3a^2x$, and $\frac{3b^2}{5cx^4y}$.

3. Reduce $\frac{a-x}{a^2-x^2}$, $\frac{ax+x^2}{ab^2+b^2x}$, and $\frac{2x^3-16x-6}{3x^3-24x-9}$, to their lowest terms. *Results* (not in order), $\frac{x}{b^2}$, $\frac{2}{3}$, and $\frac{1}{a+x}$.

4. Reduce $\frac{n^2-2n+1}{n^2-1}$, $\frac{3a^3-3ab^2}{5ab+5b^2}$, and $\frac{x^3-b^2x}{x^2+2bx+b^2}$, to their lowest terms. *Results*, $\frac{x^2-bx}{x+b}$, $\frac{n-1}{n+1}$, and $\frac{3a^2-3ab}{5b}$.

5. Reduce $\frac{x^2-2xy+y^2}{x^2-y^2}$ to its lowest terms.

6. Reduce $\frac{9x^2y+9xy^2}{3x^2+6xy+3y^2}$ to its lowest terms.

7. Reduce $\frac{5n^2-10n+5}{7n^2-7}$ to its lowest terms.

Result, $\frac{5\,(n-1)}{7\,(n+1)}$.

8. Reduce $\frac{y^3-x^3}{y^2-2xy+x^2}$ to its lowest terms.

Result, $\frac{y^2+xy+x^2}{y-x}$.

9. Reduce $\frac{n^3-2n^2}{n^2-4n+4}$ to its lowest terms.

10. Reduce $\frac{a^6-x^6}{a^4-x^4}$ to its lowest terms.

Result, $\frac{a^4+a^2x^2+x^4}{a^2+x^2}$.

Scholium 1.—Whenever the common factors of the term are not readily discernible, use the process for finding their H. C. D. (**137**).

11. Reduce $\frac{x^2-a^2}{x^4-a^4}$, $\frac{a^3-b^3}{a-b}$, $\frac{x^3+y^3}{x^2-y^2}$, and $\frac{a^3-b^3}{a^4-a^2b^2}$ to their lowest terms.

Results, a^2+ab+b^2, $\frac{x^2-xy+y^2}{x-y}$, $\frac{a^2+ab+b^2}{a^3+a^2b}$, and $\frac{1}{x^2+a^2}$.

12. Reduce $\frac{6x^2-7x-20}{4x^3-27x+5}$, $\frac{3ax^2-10ax+3a}{5a^2x^2-5a^2x-30a^2}$, and $\frac{24x^4-22x^2+5}{48x^4+16x^2-15}$ to their lowest terms.

Results, $\frac{2x^2-1}{4x^2+3}$, $\frac{3x-1}{10a+5ax}$, and $\frac{3x+4}{2x^2+5x-1}$.

Scholium 2.—The opposite process is sometimes serviceable, viz.: the introduction of a factor into both terms of a fraction which will give it a more convenient form.

13. What factor will change $\frac{x+1}{x^2+x+1}$ to $\frac{x^2-1}{x^3-1}$?

Ans., $x-1$.

14. What factor will change $\frac{a^2-ab+b^2}{a-b}$ to $\frac{a^3+b^3}{a^2-b^2}$?

Ans., $a+b$.

15. What factor will change $\frac{a^2-2ax+4x^2}{a^3-2a^2x+4ax^2-8x^3}$ to $\frac{a^3+8x^3}{a^4-16x^4}$?

Ans., $a+2x$.

16. What factor will change $\frac{a^3-2a^2b+4ab^2-8b^3}{a^2-2ab+4b^2}$ to $\frac{a^4-16b^4}{a^3+8b^3}$?

[**Note.**—It requires no special ingenuity to solve such problems, since, if the factor does not readily appear, it can be found by dividing a term of one fraction by the corresponding term of the other.]

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160. **Prob. 2.—To reduce a fraction from an improper to an integral or mixed form.**

Rule.—*Perform the division indicated.* **(141.)**

Demonstration.—The operation is explained in the same manner as the corresponding case in division.

EXAMPLES.

Ex. 1. Reduce $\frac{10ax+c-b}{2x}$ to an integral or mixed form.

Model Solution.—This being an indicated operation in Division, I have but to perform the division. Now, since the sum of the quotients is equal to the quotient of the sum, I have but to divide each term of $10ax+c-b$ by $2x$, or divide such as I can and indicate the division of the others, and add the results (**112**). Thus I find that $\frac{10ax+c-b}{2x}=5a+\frac{c-b}{2x}$.

2. Reduce $\frac{3a^2-9ac+x-6a}{3a}$, $\frac{5xy+ab+x}{x}$, and $\frac{15a^2-4a+6}{5a}$ to integral or mixed forms.

Results, $3a-\frac{4a-6}{5a}$, $5y+1+\frac{ab}{x}$, and $a-3c-2+\frac{x}{3a}$.

3. Reduce $\frac{x^2+12x+18}{x+3}$, $\frac{a^2+4ab+4b^2+c}{a+2b}$, $\frac{x^3-y^3}{x+y}$, and $\frac{a^3}{a+x}$ to integral or mixed forms.

Results, $x^2-xy+y^2-\frac{2y^3}{x+y}$, $x+9-\frac{9}{x+3}$, or $6+2x-\frac{x^2}{x+3}$, and $a+2b+\frac{c}{a+2b}$.

4. Reduce $\frac{a^3-1}{a-1}$, $\frac{x^5-y^5}{x-y}$, $\frac{a^3+3a^2b+3ab^2+b^3}{a^2+2ab+b^2}$, and $\frac{x^4-y^4}{x+y}$ to integral or mixed forms.

Results, $a+b$, $x^4+x^3y+x^2y^2+xy^3+y^4$, $x^3-x^2y+xy^2-y^3$, and a^2+a+1.

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161. Cor.—*By means of negative indices (exponents) any fraction can be expressed in the integral form.*

Thus $\frac{m}{x}=m\times\frac{1}{x}$; and by (**43**) $\frac{1}{x}=x^{-1}$. $\therefore\ \frac{m}{x}=mx^{-1}$.

5. Express $\frac{a^2b}{x^2y^3}$, and $\frac{a-b}{mx^3}$ in the integral form.

6. Show that $\frac{a+b}{m(a+b)^{-2}}=m^{-1}a^3+3m^{-1}a^2b+3m^{-1}ab^2+m^{-1}b^3$.

7. Express $\frac{7x^5y^2}{8x^6y}$, $\frac{(x+y)^2}{(x+y)^5}$, $\frac{x^5y^3}{a^{-2}x^{-3}y^2}$, and $\frac{x^2(m+n)}{(m-n)^{-2}}$ in the integral form.

Results, $(x+y)^{-3}$, a^2x^8y, $m^3x^2-m^2nx^2-mn^2x^2+n^3x^2$, and $7\times 8^{-1}x^{-1}y$.

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162. Prob. 3.—To reduce numbers from the integral or mixed to the fractional form.

Rule.—*Multiply the integral part by the given denominator, and annexing the numerator of the fractional part, if any, write the sum over the given denominator.*

Demonstration.—In the case of a number in the integral form, the process consists of multiplying the given number by the given denominator and indicating the division of the product by the same number, and hence is equivalent to multiplying and dividing by the same quantity, which does not change the value of the number. The same is true as far as relates to the integral part of a mixed form, after which the two fractional parts are to be added together. As they have the same divisors, the dividends can be added upon the principle that the sum of the quotients equals the quotient of the sum (**103**).

EXAMPLES.

Ex. 1. Reduce $2a-x^2+\frac{3ax-4a^2}{a-x}$ to a fractional form.

Model Solution.—Multiplying $2a-x^2$ by $a-x$, I have $2a^2-ax^2-2ax+x^3$. Now indicating the division of this result by $2a-x^2$, I have $\frac{2a^2-ax^2-2ax+x^3}{a-x}$. $\therefore 2a-x^2+\frac{3ax-4a^2}{a-x}=\frac{2a^2-ax^2-2ax+x^3}{a-x}+\frac{3ax-4a^2}{a-x}$. But, as the sum of these two quotients equals the quotient of the sum, I have, after uniting similar terms, $\frac{x^3-ax^2+ax-2a^2}{a-x}$.

2. Reduce $a-b+\frac{b^2}{a+b}$ to the form of a fraction.

Result, $\frac{a^2}{a+b}$.

3. Reduce $1+\frac{2x}{y-x}$ to the form of a fraction.

Result, $\frac{y+x}{y-x}$.

4. Reduce $a+b-\frac{a^2-b^2+b}{a-b}$ to the form of a fraction.

Result, $-\frac{b}{a-b}$, or $\frac{b}{b-a}$.

5. Reduce $x+2+\frac{x^2+4x+4}{x-2}$ to the form of a fraction.

Result, $\frac{2x^2+4x}{x-2}$.

6. Reduce $5x-\frac{2x-5}{3}$ to the form of a fraction.

Result, $\frac{13x+5}{3}$.

7. Reduce $x+1+\frac{x+1}{x}$ to the form of a fraction.

Result, $\frac{(x+1)^2}{x}$.

8. Reduce $2a-3b+4c+\frac{2a^2bc+9ab^2c-12abc^2}{3abc}$. *Res.*, $\frac{8}{3}a$.

9. Reduce $x-\frac{2abx-2b^2x}{a^2-b^2}$ to a fraction. *Res.*, $\frac{x(a-b)}{a+b}$.

10. Reduce $x+y-\frac{x^2-y^2+7}{x-y}$. *Result*, $\frac{7}{y-x}$.

11. Reduce $x^2+2xy+y^2-\frac{x^3-3x^2y+3xy^2-y^3}{x+y}$.

Result, $\frac{2y(3x^2+y^2)}{x+y}$.

12. Reduce $\frac{a^9x}{a^2-x^2}-(a^5x^3+a^7x)$. *Result*, $\frac{a^5x^5}{a^2-x^2}$.

13. Reduce $3a-9-\frac{3a^2-30}{a+3}$. *Result*, $\frac{3}{a+3}$.

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163. **Prob. 4.—To reduce fractions having different denominators to equivalent fractions having a common denominator.**

Rule.—*Multiply both terms of each fraction by the denominators of all the other fractions.*

Demonstration.—This gives a common denominator, because each denominator will then be the product of all the denominators of the several fractions. The value of any one of the fractions is not changed, because both numerator and denominator are multiplied by the same number (**100**).

EXAMPLES.

Ex. **1.** Reduce the fractions $\frac{x}{y}$, $\frac{2-b}{a+b}$, and $\frac{3a-x}{a-b}$ to equivalent fractions having a common denominator.

Model Solution.—Multiplying both terms of the fraction $\frac{x}{y}$ by $a+b$ and $a-b$, or by a^2-b^2, I have $\frac{a^2x-b^2x}{a^2y-b^2y}$ which has the same value as $\frac{x}{y}$, since the numerator and denominator have been multiplied by the same number. In like manner multiplying both terms of $\frac{2-b}{a+b}$, by y and $a-b$, I have $\frac{2ay-aby-2by+b^2y}{a^2y-b^2y}$, the value of which is the same as $\frac{2-b}{a+b}$, since, etc. Finally, multiplying both terms of $\frac{3a-x}{a-b}$ by y and $a+b$, I have $\frac{3a^2y-axy+3aby-bxy}{a^2y-b^2y}$, which has the same value as $\frac{3a-x}{a-b}$, since, etc. These fractions have the common denominator a^2y-b^2y, as in each case the new denominator is the product of all the old ones.

2. Reduce $\frac{3x}{2y}$, $\frac{5b}{2c}$, $\frac{3y}{7x}$, and $\frac{m}{n^2}$ to forms having a C. D.

Queries.—By what are both terms of $\frac{3x}{2y}$ to be multiplied? By what both terms of $\frac{5b}{2c}$? By what both terms of $\frac{3y}{7x}$?

Results, $\frac{42cn^2x^2}{28cn^2xy}$, $\frac{70bn^2xy}{28cn^2xy}$, $\frac{12cn^2y^2}{28cn^2xy}$, and $\frac{28cmxy}{28cn^2xy}$.

3. Reduce $\frac{x}{3}$, $\frac{x+1}{5}$, and $\frac{1-x}{1+x}$ to forms having a C. D.

Results, $\frac{5x+5x^2}{15+15x}$, $\frac{3+6x+3x^2}{15+15x}$, and $\frac{15-15x}{15+15x}$.

4. Reduce $\frac{a}{a+b}$, and $\frac{b}{a-b}$ to forms having a C. D.

Results, $\frac{a^2-ab}{a^2-b^2}$, and $\frac{ab+b^2}{a^2-b^2}$.

Scholium.—Practically, this method consists in multiplying all the denominators together for a new denominator, and each numerator into all the denominators except its own for a new numerator. But it is much better to repeat the rule as given above, and let that be the form of conception, as it keeps the *principle* constantly before the mind.

***164.* Cor.**—*To reduce fractions to equivalent ones having the Lowest Common Denominator, find the L. C. M. of all the denominators for the new denominator. Then multiply both terms of each fraction by the quotient of that L. C. M. divided by the denominator of that fraction.*

Demonstration.—The purpose in getting the L. C. M. is to get the *lowest* number which can be divided by each of the denominators. That the process does not change the value of the fractions is evident from (**100**), the same as under the general rule.

EXAMPLES.

5. Reduce $\frac{a}{1-a}$, $\frac{a^2}{(1-a)^2}$, and $\frac{a^8}{(1-a)^3}$ to equivalent fractions having the L. C. D.

Model Solution.—By inspection I observe that $(1-a)^3$ is the L. C. M. of the denominators, since it is the lowest number which contains itself, and it also contains each of the other denominators. Now, to make the denominator of $\frac{a}{1-a}$, $(1-a)^3$, I must multiply it by $(1-a)^3 \div (1-a)$; *i. e.*, by $(1-a)^2$. But to preserve the value of the fraction, I must multiply the numerator by the same quantity. Thus $\frac{a}{1-a}=\frac{a(1-a)^2}{(1-a)^3}=\frac{a-2a^2+a^3}{1-3a+3a^2-a^3}$; etc.

6. Reduce $\frac{x^2+y^2}{x-y}$, $\frac{x^2-y^2}{x+y}$ to forms having the L. C. D.

Results, $\frac{x^3-xy^2-x^2y+y^3}{x^2-y^2}$, and $\frac{x^3+x^2y+xy^2+y^3}{x^2-y^2}$.

7. Reduce $\frac{c}{x+y}$, $\frac{a}{x^3+y^3}$, $\frac{b}{x+y}$ to forms having the L. C. D.

Results, $\frac{a}{x^3+y^3}$, $\frac{b\,(x^2-xy+y^2)}{x^3+y^3}$, and $\frac{c\,(x^2-xy+y^2)}{x^3+y^3}$.

8. Reduce mn, $\frac{m-n}{m+n}$, $\frac{m+n}{m-n}$ to forms having the L. C. D.

Results, $\frac{m^3n-mn^3}{m^2-n^2}$, $\frac{(m-n)^2}{m^2-n^2}$, and $\frac{(m+n)^2}{m^2-n^2}$.

9. Reduce $\frac{1+x}{1-x}$, $\frac{1+x^2}{1-x^2}$, $\frac{1+x^3}{1-x^3}$ to forms having the L.C.D.

The L. C. D. is $1+x-x^3-x^4$.

10. Reduce $\frac{a^2+b^2}{a^2-b^2}$, and $\frac{a+b}{(a-b)^2}$ to forms having the L.C.D.

11. Reduce $\frac{1}{m+n}$, $\frac{a}{m^3+n^3}$, $\frac{b}{m+n}$ to forms having the L.C.D.

12. Reduce $\frac{2}{x}$, $\frac{3}{2x-1}$, and $\frac{2x-3}{4x^2-1}$ to forms having the L.C.D.

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165. Prob. 5.—To reduce Complex Fractions to the form of Simple Fractions.

Rule.—*Multiply numerator and denominator of the complex fraction by the product of all the denominators of the partial fractions found in them.*

Or, multiply by the L. C. M. of the denominators of the partial fractions.

Demonstration.—This process removes the partial denominators, since each fraction is multiplied by its own denominator, at least, and this is done by dropping the denominator. It does not alter the value of the fraction, since it is multiplying dividend and divisor by the same quantity.

EXAMPLES.

Ex. 1. Reduce $\dfrac{\frac{2x}{3b^2}}{\frac{4y}{5a^2}}$ to a simple fractional form.

Model Solution.

Operation. $$\frac{\frac{2x}{3b^2}}{\frac{4y}{5a^2}}=\frac{\frac{2x}{3b^2}\times 5a^2\times 3b^2}{\frac{4y}{5a^2}\times 5a^2\times 3b^2}=\frac{10a^2x}{12b^2y}=\frac{5a^2x}{6b^2y}.$$

Explanation.—In order to free the numerator of its denominator, $3b^2$, I multiply the numerator $\frac{2x}{3b^2}$ by $3b^2$; but in order that this may not change the value of the fraction, I also multiply the denominator by the same. In like manner to free the denominator $\frac{4y}{5a^2}$ of its denominator, I multiply numerator and denominator by $5a^2$. Indicating these operations I have $\dfrac{\frac{2x}{3b^2}\times 5a^2\times 3b^2}{\frac{4y}{5a^2}\times 5a^2\times 3b^2}$. To multiply $\frac{2x}{3b^2}$ by $3b^2$ I drop its denominator and have $2x$, which multiplied by $5a^2$ gives for the new numerator $10a^2x$. So, also, I obtain the new denominator by dopping $5a^2$ and multiplying $4y$ by $3b^2$, getting thereby $12b^2y$. Therefore the simple fraction is $\frac{10a^2x}{12b^2y}$ which reduced to lowest terms is $\frac{5a^2x}{6b^2y}$.

2. Reduce $\dfrac{a+\frac{1}{c}}{m}$ to a simple form. *Result,* $\dfrac{ac+1}{cm}$.

3. Reduce $\dfrac{m}{a+\frac{1}{c}}$ to a simple form. *Result,* $\dfrac{cm}{ac+1}$.

4. Reduce $\dfrac{a+\frac{b}{c}}{x-\frac{m}{n}}$ to a simple form. *Result,* $\dfrac{acn+bn}{cnx-cm}$.

5. Reduce $\dfrac{\frac{3}{\text{[illegible]}}}{3x-\frac{a}{y}}$ to a simple form.

6. Reduce $\dfrac{5-\frac{3}{7}a}{10+\frac{5}{14}x}$ to a simple form.

7. Reduce $\dfrac{\frac{a}{b}+\frac{x}{y}}{\frac{a}{m}-\frac{x}{n}}$ to a simple form.

8. Reduce $\dfrac{a+\frac{b-a}{1+ab}}{1-\frac{ab-a^2}{1+ab}}$ to a simple form.

ADDITION

SECTION III.

166. Prob.—To add Fractions.

Rule.—*Reduce them to forms having a common denominator, if they have not such forms, and then add the numerators, and write the sum over the common denominator.*

Demonstration.—The reduction of the several fractions to forms having a common denominator, does not alter their values (**163**), and hence does not alter the sum. Then, when they have a common denominator (divisor), the sum of the several quotients is equal to the quotient arising from dividing the sum of the several dividends by the common divisor, or denominator (**103**).

EXAMPLES.

Ex. 1. What is the sum of $\frac{1+x}{1-x}$, $\frac{1+x^2}{1-x^2}$, and $\frac{1+x^3}{1-x^3}$.

Model Solution.

Operation.—The L. C. M. of $1-x$, $1-x^2$ and $1-x^3$ is $(1-x^3) \times (1+x) = 1+x-x^3-x^4$.

$$\frac{(1+x)\times(1+2x+2x^2+x^3)}{(1-x)\times(1+2x+2x^2+x^3)} = \frac{1+3x+4x^2+3x^3+x^4}{1+x-x^3-x^4}$$

$$\frac{(1+x^2)\times(1+x+x^2)}{(1-x^2)\times(1+x+x^2)} = \frac{1+x+2x^2+x^3+x^4}{1+x-x^3-x^4}$$

$$\frac{(1+x^3)\times(1+x)}{(1-x^3)\times(1+x)} = \frac{1+x+x^3+x^4}{1+x-x^3-x^4}.$$

$$\therefore \frac{1+x}{1-x} + \frac{1+x^2}{1-x^2} + \frac{1+x^3}{1-x^3} = \frac{1+3x+4x^2+3x^3+x^4}{1+x-x^3-x^4} + \frac{1+x+2x^2+x^3+x^4}{1+x-x^3-x^4} + \frac{1+x+x^3+x^4}{1+x-x^3-x^4} \qquad \text{(A.)}$$

$$= \frac{3+5x+6x^2+5x^3+3x^4}{1+x-x^3-x^4}.$$

Explanation.—Explain the reduction to a common denominator as under (**163**) unless that is already sufficiently familiar.

Having reduced the fractions to the L. C. D. I find (read A). Now since the sum of these quotients is equal to the quotient arising from dividing the sum of the several dividends, or numerators, by the common divisor, or denominator (**103**), I add the numerators and write the sum over the common denominator, which gives $\frac{3+5x+6x^2+5x^3+3x^4}{1+x-x^3-x^4}$ for the sum of $\frac{1+x}{1-x}$, $\frac{1+x^2}{1-x^2}$, and $\frac{1+x^3}{1-x^3}$.

2. Add $\frac{x}{2y}$, $\frac{c-a}{3y}$, and $\frac{c+a}{4y}$. *Sum*, $\frac{6x+7c-a}{12y}$.

3. Add $\frac{a}{b}$, $\frac{c}{ab}$, $\frac{b}{a}$, and $\frac{a}{b^2}$. *Sum*, $\frac{a^2b+cb+b^3+a^2}{ab^2}$.

4. Add $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{12}$, $\frac{x}{18}$, $\frac{x}{6}$, and $\frac{x}{9}$. *Sum*, x.

5. Add $\frac{a-3}{3}$ and $\frac{5+a}{6}$ *Sum*, $\frac{3a-1}{6}$.

6. Add $\frac{1}{1+a}$, $\frac{a}{1-a}$, and $\frac{a}{1+a}$. *Sum*, $\frac{1}{1-a}$.

7. Add $\frac{4a^2}{1-a^4}$ and $\frac{1-a^2}{1+a^2}$. *Sum*, $\frac{1+a^2}{1-a^2}$.

8. Add $\frac{1}{1+x}$ and $\frac{1}{1-x}$. *Sum*, $\frac{2}{1-x^2}$.

9. Add $\frac{1}{a^2-b^2}$ and $\frac{1}{(a-b)^2}$. *Sum*, $\frac{2a}{a^3-a^2b-ab^2+b^3}$.

10. Add $\frac{x}{x^2-y^2}$, $\frac{y}{x+y}$ and $\frac{1}{x-y}$.

Sum, $\frac{2x+xy-y^2+y}{x^2-y^2}$.

167. Cor.—*Expressions in the mixed form may either be reduced to the improper form and then added, or the integral parts may be added into one sum, and the fractional into another, and these results added.*

Ex. 1. Add $7x+\frac{x-2}{3}$ and $8x+\frac{3x+4}{5x}$.

FIRST FORM OF OPERATION.

$$7x+\frac{x-2}{3}=\frac{22x-2}{3}$$

$$8x+\frac{3x+4}{5x}=\frac{40x^2+3x+4}{5x}$$

$$\frac{(22x-2)\times 5x}{3\quad\times 5x}=\frac{110x^2-10x}{15x}$$

$$\frac{(40x^2+3x+4)\times 3}{5x\quad\times 3}=\frac{120x^2+9x+12}{15x}$$

$$\frac{110x^2-10x}{15x}+\frac{120x^2+9x+12}{15x}=\frac{230x^2-x+12}{15x}$$

SECOND FORM OF OPERATION.

$$7x+8x=15x$$

$$\frac{(x-2)\times 5x}{3\quad\times 5x}=\frac{5x^2-10x}{15x}$$

$$\frac{(3x+4)\times 3}{5x\quad\times 3}=\frac{9x+12}{15x}$$

$$\therefore\ 7x+\frac{x-2}{3}+8x+\frac{3x+4}{5x}=15x+\frac{5x^2-10x}{15x}+\frac{9x+12}{15x}=15x+\frac{5x^2-x+12}{15x}.$$

Explanation.—Since the sum of several numbers is the same in whatever order their parts are added, I may add the integral parts first. Adding $7x$ and $8x$ I have $15x$. Reducing the fractions to forms having a common denominator, $\frac{x-2}{3}$ becomes $\frac{5x^2-10x}{15x}$, and $\frac{3x+4}{5x}$ becomes $\frac{9x+12}{15x}$. Adding these I have $\frac{5x^2-x+12}{15x}$ which added to $15x$, the sum of the integral parts, gives for the entire sum $15x+\frac{5x^2-x+12}{15x}$.

2. Add $2x$, $3x+\frac{2a}{5}$, and $x+\frac{8a}{9}$. *Sum*, $6x+\frac{58a}{45}$.

3. Add $a-\frac{3x^2}{b}$ to $b+\frac{2ax}{c}$. *Sum*, $a+b+\frac{2abx-3cx^2}{bc}$.

4. Add $7x+\frac{x-2}{3}$ and $9x-\frac{2x-3}{5x}$.
Sum, $16x+\frac{5x^2-16x+9}{15x}$.

5. Add $6x-\frac{3x}{4}$, $\frac{2x}{5}-8x$, and $3x-\frac{x}{2}$. *Sum*, $x-\frac{17x}{20}$.

6. Add $3mx-\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}-2mx+4$.
Sum, $4+mx-\frac{4ab}{a^2-b^2}$.

7. Add $6x^{\frac{2}{3}}y^{\frac{1}{2}}-3x-\frac{x-y}{x^2+y^2}$ and $5x-2x^{\frac{2}{3}}y^{\frac{1}{2}}+\frac{x}{x^2-y^2}$.
Sum, $4x^{\frac{2}{3}}y^{\frac{1}{2}}+2x+\frac{2xy^2+x^2y-y^3}{x^4-y^4}$.

8. Add $\frac{p-q}{pq}$, $\frac{r-p}{pr}$, and $\frac{q-r}{qr}$. *Sum*, 0.

9. Add $\frac{a}{a-x}$, $\frac{3a}{a+x}$, and $-\frac{2ax}{a^2-x^2}$. *Sum*, $\frac{4a}{a+x}$.

10. Add $\frac{a+b}{(b-c)(c-a)}$, $\frac{b+c}{(c-a)(a-b)}$, and $\frac{c+a}{(a-b)(b-c)}$.
Sum, 0.

Suggestion.—The L. C. D. is $(a-b)(b-c)(c-a)$.

Subtraction

SECTION IV.

168. Prob.—To subtract Fractions.

Rule.—*Reduce the fractions to forms having a common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and place the remainder over the common denominator.*

Demonstration.—Since the values of the fractions are not altered by reducing them to forms having a common denominator, their difference is not altered. Now subtracting the numerator of the subtrahend from the numerator of the minuend, and dividing the difference by the common denominator, gives the difference between the fractions, since *the difference of the quotients of two quantities divided by a common divisor, is the same as the quotient of the difference divided by the same divisor* (**104**).

EXAMPLES.

Ex. 1. From $\frac{x+y}{x-y}$ subtract $\frac{x-y}{x+y}$.

Model Solution.

Operation. $(x-y)(x+y)=x^2-y^2$

$$\frac{(x+y)\times(x+y)}{(x-y)\times(x+y)}=\frac{x^2+2xy+y^2}{x^2-y^2}$$

$$\frac{(x-y)\times(x-y)}{(x+y)\times(x-y)}=\frac{x^2-2xy+y^2}{x^2-y^2}$$

$$(x^2+2xy+y^2)-(x^2-2xy+y^2)=4xy$$

$$\therefore \frac{x+y}{x-y}-\frac{x-y}{x+y}=\frac{4xy}{x^2-y^2}$$

Explanation.—The L. C. M. of $x-y$ and $x+y$ is their product, since they have no common factor. Hence x^2-y^2 is the L. C. D. To reduce $\frac{x+y}{x-y}$ to a form having this denominator, I multiply both its terms by $x+y$, which gives $\frac{x^2+2xy+y^2}{x^2-y^2}$. In like manner multiplying both terms of $\frac{x-y}{x+y}$ by $x-y$, I have $\frac{x^2-2xy+y^2}{x^2-y^2}$. I have now to subtract $\frac{x^2-2xy+y^2}{x^2-y^2}$ from $\frac{x^2+2xy+y^2}{x^2-y^2}$. Since the difference of the quotients of two numbers divided by the same number, is the same as the quotient arising from dividing the difference between those numbers by the common divisor, I take the difference of the numerators (the quantities to be divided) which is $4xy$, and dividing it by x^2-y^2 I have $\frac{4xy}{x^2-y^2}$, for the remainder of $\frac{x+y}{x-y}$ less $\frac{x-y}{x+y}$.

2. From $\frac{1+x}{1-x}$ take $\frac{1-x}{1+x}$. *Remainder*, $\frac{4x}{1-x^2}$.

3. From $\frac{1}{a-x}$ take $\frac{1}{a+x}$. *Remainder*, $\frac{2x}{a^2-x^2}$.

4. From $\frac{x}{x-3}$ take $\frac{x+3}{x}$. *Remainder*, $\frac{9}{x^2-3x}$.

5. From $3x$ take $\frac{3a+12x}{5}$. *Remainder*, $\frac{3x-3a}{5}$

6. From $9y$ take $\frac{1+2y}{8}$. *Remainder*, $\frac{70y-1}{8}$

7. From $\frac{1}{a-b}$ take $\frac{1}{a^2-2ab+b^2}$. *Rem.*, $\frac{a-b-1}{(a-b)^2}$.

8. From $\frac{1}{x+4}$ take $\frac{2-x^2-3x}{x^2+10x+24}$. *Rem.*, $\frac{(x+2)^2}{x^2+10x+24}$.

9. From $\frac{3x+2}{a}$ take $\frac{7ax-10a}{a^2}$ *Remainder*, $\frac{4(3-x)}{a}$

10. Combine the following fractions $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x-x^2}{x^2-4}$.

Result, $\frac{1}{x+2}$.

11. Combine $\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-c)(b-a)} + \frac{1}{c(c-a)(c-b)}$.

Result, $\frac{1}{abc}$.

12. Combine $\frac{3a-4b}{7} - \frac{2a-b-c}{3} + \frac{15a-4c}{12} - \frac{a-4b}{21}$.

Result, $\frac{81a-4b}{84}$.

13. Combine $\frac{1}{a+b} + \frac{b}{a^2-b^2} - \frac{a}{a^2+b^2}$.

Result, $\frac{2ab^2}{a^4-b^4}$.

14. Combine $\frac{3}{1-2x} - \frac{7}{1+2x} - \frac{4-20x}{4x^2-1}$. *Result,* 0.

15. Combine $\frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$.

Result, $\frac{2x-3}{(x^2-1)(2x+3)}$.

16. Combine $\frac{1}{x^2-y^2} + \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2}$.

Result, $\frac{x^2-4xy-y^2}{(x^2-y^2)^2}$.

169. **Cor.**—*Mixed numbers may be subtracted by annexing the subtrahend with its signs changed, to the minuend, and then combining the terms. The reason for the change of signs is the same as in whole numbers* (77).

EXAMPLES.

17. From $3x + \frac{x+y}{m}$ take $x - \frac{x-y}{m}$.

Remainder, $2x + \frac{2x}{m}$.

18. From $x - \frac{4a-b}{2}$ subtract $7x - \frac{3a-2b}{3}$.

Remainder, $-\frac{b}{6} - 6x - a$.

19. From $3a$ take $\frac{3x+12a}{5}$. *Remainder,* $\frac{3a-3x}{5}$.

20. From $2x + \frac{5x-2}{7}$ take $3x - \frac{4x+5}{6}$.

Remainder, $\frac{16x+23}{42}$.

Multiplication

SECTION V.

170. **Prob. 1.—To multiply a Fraction by an Integer.**

Rule. — *Multiply the numerator or divide the denominator.*

Demonstration.—Since numerator is dividend and denominator divisor, and the value of the fraction is the quotient, this rule is a direct consequence of (**101, 102**).

EXAMPLES.

Ex. 1. Multiply $\frac{m-n}{3xy}$ by $m+n$.

Model Solution.

Operation. $\frac{m-n}{3xy} \times (m+n) = \frac{m^2-n^2}{3xy}$.

Explanation.—Since the value of a fraction is the quotient of the numerator divided by the denominator, and multiplying the dividend multiplies the quotient, I multiply the numerator of $\frac{m-n}{3xy}$ by $m+n$, and have $\frac{m^2-n^2}{3xy}$, which is therefore $m+n$ times $\frac{m-n}{3xy}$.

2. Multiply $\frac{2mx}{3a^2b^2}$ by $3a$.

Model Solution.

Operation. $\frac{2mx}{3a^2b^2} \times 3a = \frac{2mx}{ab^2}$.

Explanation.—Since $2mx$ is to be divided by $3a^2b^2$ if I divide the divisor by $3a$, thus making it $3a$ times as small, it will go into the dividend $3a$ times as many times as before. Hence $\frac{2mx}{ab^2}$ is $3a$ times $\frac{2mx}{3a^2b^2}$. The operation is performed by cancellation.

3. Multiply $\frac{x-y}{a-1}$ by $x+y$.

4. Multiply $\frac{ac}{b(x-y)}$ by $x-y$. Product.

5. Multiply $\frac{m}{a^5-a}$ by a^2+1. *Prod.*, $\frac{m}{a^3-a}$.

Suggestion. $a^5-a = a(a^4-1) = a(a^2-1)(a^2+1)$.

6. Multiply $\frac{(a+y)^2}{m(a^2-y^2)}$ by $(a+y)^2$.

Suggestions.—Multiply by one of the factors of $(a+y)^2$ by rejecting it from the denominator, giving $\frac{(a+y)^2}{m(a-y)}$, and this product by the other factor, giving $\frac{(a+y)^3}{am-my}$.

7. Multiply $\frac{3}{a-x}$ by a^2-x^2. *Prod.*, $3a+3x$.

8. Multiply $\frac{6ax^2}{a^2-y^2}$ by $2a+2y$. *Prod.*, $\frac{12ax^2}{a-y}$.

9. Multiply $\frac{3a-2y}{3b}$ by $3b$. *Prod.*, $3a-2y$.

Simply remove the denominator. Why?

10. Multiply $\frac{3ax}{x-y}$ by $x-y$.

11. Multiply $\frac{3ax}{x-y}$ by $2ax\,(x-y)$. *Prod.*, $6a^2x^2$.

12. Multiply $\frac{3c}{x-1}$ by x^2-1.

13. Multiply $\frac{x+y}{x-y}$ by $x^2-2xy+y^2$.

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171. Prob. 2.—To multiply by a Fraction.

Rule.—*Multiply by the numerator and divide by the denominator.*

Demonstration.—Let it be required to multiply m, which is either an integer or a fraction, by $\frac{a}{b}$.

1st. Suppose a and b are both integers. Multiplying m by a gives a product b times too large, since we were to multiply by only a bth part of a; hence we divide the product, am, by b, and have $\frac{am}{b}$.

2nd. When either a or b, or both, are fractions. Let c be the factor by which numerator and denominator of $\frac{a}{b}$ must be multiplied to make $\frac{a}{b}$ a simple fraction (**165**). Then will $\frac{ac}{bc}$ be a simple fraction, *i. e.*, ac and bc are each integral; and the multiplication is effected as in Case 1st, giving $\frac{acm}{bc}$. This reduced by dividing both terms by c, *i. e.*, by cancelling c, gives $\frac{am}{b}$. Hence we see that in *any* case, to multiply by a fraction, we have only to multiply the multiplicand by the numerator of the multiplier, and divide this product by the denominator. It is also to be observed that this reasoning applies equally well whether the *multiplicand* is integral or fractional.

EXAMPLES.

Ex. 1. Multiply $\frac{(a-b)^2}{x^3-y^3}$ by $\frac{x-y}{a-b}$.

Model Solution.

Operation. $\frac{(a-b)^2}{x^3-y^3}\times(x-y)=\frac{(a-b)^2}{x^2+xy+y^2}$

Again, $\frac{(a-b)^2}{x^2+xy+y^2}\div(a-b)=\frac{a-b}{x^2+xy+y^2}$.

$$\therefore\ \frac{(a-b)^2}{x^3-y^3}\times\frac{x-y}{a-b}=\frac{a-b}{x^2+xy+y^2}.$$

Explanation.—In order to multiply $\frac{(a-b)^2}{x^3-y^3}$ by $\frac{x-y}{a--b}$, I first multiply by $x-y$. This is effected by dividing the denominator, x^3-y^3, by $x-y$, (**102**), and gives $\frac{(a-b)^2}{x^2+xy+y^2}$. But, since this multiplier was to be divided by $a-b$, the product now obtained must be divided by the same. Dividing $\frac{(a-b)^2}{x^2+xy+y^2}$ by $a-b$ by dividing the numerator (**101**), I have for the complete product $\frac{a-b}{x^2+xy+y^2}$.

Practically, the operation is performed by cancelling $x-y$ from x^3-y^3, leaving x^2+xy+y^2; and $a-b$ from $(a-b)^2$, leaving $a-b$.

2. Multiply $\frac{3m}{cx}$ by $\frac{c}{3}$. *Prod.*, $\frac{m}{x}$.

3. Multiply $\frac{2a^2}{41b^3x}$ by $\frac{11x^3b}{5a^2y}$. *Prod.*, $\frac{22x^2}{205b^2y}$.

4. Multiply $-\frac{3cx}{5ay}$ by $\frac{2c}{3y^3}$. *Prod.*, $-\frac{2c^2x}{5ay^4}$.

5. Multiply $-\frac{2ax}{3by}$ by $-\frac{5a^2}{7by^2}$. *Prod.*, $\frac{10a^3x}{21b^2y^3}$.

Scholium.—When there are no common factors in the numerators and denominators of the fractions to be multiplied together, the process consists simply in multiplying numerators together for a new numerator and denominators for a new denominator.

6. Multiply $4x^2-9y^2$ by $\frac{2m}{2x-3y}$.

Suggestion.—In this case divide first, obtaining $2x+3y$. Multiply this by $2m$.

Prod., $4mx+6my$.

7. Multiply $\frac{x}{a}$ by $\frac{x+a}{a+c}$. *Prod.*, $\frac{x^2+ax}{a^2+ac}$.

8. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. *Prod.*, $\frac{x^4-b^4}{b^2c+bc^2}$.

9. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$. *Prod.*, $\frac{a^2+b^2}{(a+b)^2}$.

Suggestion.—In the last example both operations are performed upon the denominator of the multiplicand.

10. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$. *Prod.*, $\frac{3ax-5a}{4x^2-6}$.

11. Multiply $\frac{a+x}{30}$ by $\frac{5a}{3(a+x)}$. *Prod.*, $\frac{a}{18}$.

12. Multiply $\frac{a^2-x^2}{\cancel{2}y}$ by $\frac{\cancel{2}a}{a+x}$. *Prod.*, $\frac{a^2-ax}{y}$.

13. Multiply $\frac{(a+b)^5}{(x+y)^4}$ by $\frac{(x+y)^5}{(a+b)^6}$. *Prod.*, $\frac{x+y}{a+b}$.

14. Multiply $\frac{x^n}{y^m}$ by $\frac{x^m}{y^n}$. *Prod.*, $\frac{x^{m+n}}{y^{m+n}}$.

15. Multiply $\frac{x^n}{y^n}$ by $\frac{y^m}{x^m}$. *Prod.*, $\frac{x^{n-m}}{y^{n-m}}$.

16. Multiply $\frac{a^2-b^2}{a^4-b^2}$ by $\frac{a^2+b}{a-b}$. *Prod.*, $\frac{a+b}{a^2-b}$.

17. Multiply $\frac{3c^{-2}x^3}{5a^5y^{-\frac{2}{3}}}$ by $\frac{2c^3x}{7a^{-3}y^{-\frac{1}{3}}}$, and write the result without negative exponents.

Prod., $\frac{6cx^4y}{35a^2}$.

18. Multiply together $-\frac{a}{x^2-y^2}$, $-\frac{(x+y)^2}{a^5}$, and $-\frac{3m}{(x+y)}$.

Prod., $-\frac{3m}{a^4(x-y)}$.

19. Multiply $-\frac{x^2+3x+2}{x^2+2x+1}$ by $\frac{x^2+5x+4}{x^2+7x+12}$.

Prod., $-\frac{x+2}{x+3}$.

***172.* Cor.**—*To multiply mixed numbers first reduce them to improper fractions.*

20. Multiply $1-\frac{2x}{3y}$ by $\frac{x}{y}-2$.

Prod., $\frac{7xy-2x^2-6y^2}{3y^2}$, or $\frac{x(7y-2x)}{3y^2}-2$.

21. Multiply together $-\frac{1-x^2}{1+y}$, $\frac{1-y^2}{x^2+x}$, and $1+\frac{x}{1-x}$.

Prod., $\frac{y-1}{x}$.

22. Multiply $a-\frac{x^2}{a}$ by $\frac{a}{x}+\frac{x}{a}$. *Prod.*, $\frac{a^4-x^4}{a^2x}$.

23. Multiply $1-\frac{x-y}{x+y}$ by $2+\frac{2y}{x-y}$. *Prod.*, $\frac{4xy}{x^2-y^2}$.

24. Multiply x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$.

Prod., $x^2+1+\frac{1}{x^2}$.

Operation.

$$\begin{array}{rrrrr}
 & x^2-x+1 & & & \\
 & \frac{1}{x^2}+\frac{1}{x}+1 & & & \\
\hline
 & & 1 & -\frac{1}{x} & +\frac{1}{x^2} \\
 & x & -1 & +\frac{1}{x} & \\
x^2 & -x & +1 & & \\
\hline
x^2 & & +1 & & +\frac{1}{x^2}, \text{ Product.}
\end{array}$$

DIVISION

SECTION VI.

173. Prob. 1.—To divide a Fraction by an Integer.

Rule. — *Divide the numerator or multiply the denominator.*

Demonstration.—Since numerator is dividend, and denominator divisor, and the value of the fraction the quotient, this rule is a direct consequence of (**101, 102**).

EXAMPLES.

Ex. 1. Divide $\frac{3(a^2-x^2)}{2mx-1}$ by $a-x$.

Solution.—Since the value of this fraction is the quotient of $3(a^2-x^2)$ divided by $2mx-1$, if I divide the dividend, $3(a^2-x^2)$, by $a-x$, I divide the fraction. Hence $\frac{3(a^2-x^2)}{2mx-1} \div (a-x) = \frac{3a+3x}{2mx-1}$.

2. Divide $\frac{3m-4xy}{a-b}$ by $a+b$.

Solution.—Since the value of this fraction is the quotient of $3m-4xy$ divided by $a-b$, if I multiply the divisor, $a-b$, by $a+b$, I divide the fraction. Hence $\frac{3m-4xy}{a-b} \div (a+b) = \frac{3m-4xy}{a^2-b^2}$.

3. Divide $\frac{3m^2x}{4a^2b}$ by $3x$.

4. Divide $\frac{15a^2-15x^2}{x-y}$ by $5(a+x)$. *Quot.*, $\frac{3(a-x)}{x-y}$.

5. Divide $\frac{x^2-1}{x^2+3}$ by $x-1$.

6. Divide $\dfrac{x^3-y^3}{x+y}$ by $x-y$.

7. Divide $\dfrac{x^3-y^3}{x+y}$ by $x+y$.

8. Divide $\dfrac{m}{x^2-4}$ by x^2+4.

9. Divide $\dfrac{x^5-y^5}{x+y}$ by x^2-y^2.

Quot., $\dfrac{x^4+x^3y+x^2y^2+xy^3+y^4}{(x+y)^2}$.

Suggestion.—Perform this example by dividing successively by the factors of x^2-y^2, viz., $x+y$ and $x-y$. To divide by $x-y$, divide the numerator; to divide by $x+y$, multiply the denominator.

10. Divide $\dfrac{3m^{\text{[illegible]}}y^3}{5n^{\frac{1}{2}}x^{\frac{2}{3}}}$ by $5mn^{\frac{1}{2}}x^{\frac{1}{3}}$. *Quot.*, $\dfrac{3my^3}{25nx}$.

11. Divide $\dfrac{a^2-2ab+b^2}{a+b}$ by a^2-b^2.

12. Divide $\dfrac{30\,(a^3-b^3)}{7\,(a+b)}$ by $42\,(a-b)^2$.

174. Prob. 2.—To divide by a Fraction.

Rule.—*Divide by the numerator and multiply the quotient by the denominator.*

Or, what is the same thing, invert the terms of the divisor and proceed as in multiplication.

Demonstration.—The correctness of the first process appears from the fact that division is the reverse of multiplication, which requires that we multiply by the numerator and divide by the denominator.

The process of inverting the divisor and then multiplying by it is seen to be the same as the other, since this also multiplies the dividend by the denominator of the divisor and divides by the numerator.

Again, this process may be demonstrated thus: Inverting the divisor shows how many times it is contained in 1. Then if the given divisor is contained so many times in 1, it will be contained in n, n times as many times; or in any dividend as many times the number of times it is contained in 1, as is expressed by that dividend, whether it be integral, fractional or mixed.

Scholium 1.—Since to multiply one fraction by another we may multiply the numerators together for the numerator and the denominators for the denominator, and since division is the reverse, we may perform division by dividing the numerator of the dividend by the numerator of the divisor, and the denominator of the dividend by the denominator of the divisor.

This method will coincide with the others when *they* are worked by performing the operations by division as far as practicable, and *this* is worked by performing the multiplications equivalent to the divisions when the latter are not practicable.

EXAMPLES.

Ex. 1. Divide $\frac{2y^2}{x^3+y^3}$ by $\frac{y}{x+y}$.

Model Solutions.

OPERATION BY THE FIRST METHOD.

$$\frac{2y^2}{x^3+y^3} \div y = \frac{2y}{x^3+y^3};$$

$$\frac{2y}{x^3+y^3} \times (x+y) = \frac{2y}{x^2-xy+y^2}$$

Explanation.—I first divide $\frac{2y^2}{x^3+y^3}$ by y, by dividing the numerator (**101**). But the given divisor is y *divided* by $x+y$; and as dividing the divisor multiplies the quotient (**102**), I must multiply this quotient, $\frac{2y}{x^3+y^3}$ by $x+y$. Performing this by dividing the denominator, I have for the true quotient $\frac{2y}{x^2-xy+y^2}$.

OPERATION BY THE SECOND METHOD.

$$\frac{2y^2}{x^3+y^3} \div \frac{y}{x+y} = \frac{2y^2}{x^3+y^3} \times \frac{x+y}{y} = \frac{2y}{x^2-xy+y^2}.$$

Explanation.—By inverting the divisor and indicating the multiplication of the dividend by it, I indicate that the dividend is to be multiplied by $x+y$ and divided by y, which are the operations required. In this instance I perform the multiplication by $x+y$, by dividing the denominator, and the division by y, by dividing the numerator.

The operation by the third method is of the same form as the last.

EXPLANATION BY THE THIRD METHOD.

I am to find how many times $\frac{y}{x+y}$ is contained in $\frac{2y^2}{x^3+y^3}$. Now $\frac{y}{x+y}$ is contained in 1, $\frac{x+y}{y}$ times; since y is contained in 1, $\frac{1}{y}$ times, it is contained in $x+y$, $x+y$ times $\frac{1}{y}$, or $\frac{x+y}{y}$ times. Hence if $\frac{y}{x+y}$ is contained in 1, $\frac{x+y}{y}$ times, it is contained in $\frac{2y^2}{x^3+y^3}$, $\frac{2y^2}{x^3+y^3}$ times $\frac{x+y}{y}$.

2. Divide $\frac{ax-x^2}{(a+x)^2}$ by $\frac{x^2}{a^2-x^2}$. *Quot.*, $\frac{(a-x)^2}{x\,(a+x)}$.

3. Divide $\frac{4\,(a^2-ab)}{b\,(a+b)}$ by $\frac{6ab}{a^2-b^2}$. *Quot.*, $\frac{2\,(a-b)^2}{3b^2}$.

4. Divide $\frac{a^3+3a^2x+3ax^2+x^3}{x^3-y^3}$ by $\frac{(a+x)^2}{x^2+xy+y^2}$.

Quot., $\frac{a+x}{x-y}$.

5. Divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$. *Quot.*, $\frac{3x+1}{12x}$.

6. Divide $\frac{4x}{2x-1}$ by $\frac{2}{x+1}$. *Quot.*, $\frac{2x\,(x+1)}{2x-1}$.

7. Divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+b}$. *Quot.*, $\frac{8a^2+6ab+b^2}{9a^2-4b^2}$.

8. Divide $\frac{a^2-b^2}{a^2-2ab+b^2}$ by $\frac{a^2+ab}{a-b}$. *Quot.*, $\frac{1}{a}$.

9. Divide $\frac{4a(a^2-x^2)}{3b(c^2-x^2)}$ by $\frac{a^2-ax}{bc+bx}$. *Quot.*, $\frac{4(a+x)}{3(c-x)}$.

10. Divide $x+\frac{2x}{x-3}$ by $x-\frac{2x}{x-3}$. *Quot.*, $\frac{x-1}{x-5}$.

11. Divide $x^4-\frac{1}{x^4}$ by $x-\frac{1}{x}$.

Suggestion.—This quotient should be written by inspection in the same manner as $(x^4-y^4)\div(x-y)$, and is $x^3+x+\frac{1}{x}+\frac{1}{x^3}$. Or it may be performed as follows:

$$\begin{array}{l|l}
x^4-\frac{1}{x^4} & x-\frac{1}{x} \\ \hline
\underline{x^4-x^2} & x^3+x+\frac{1}{x}+\frac{1}{x^3} \\
x^2-\frac{1}{x^4} & \\
\underline{x^2-1} & \\
\qquad 1-\frac{1}{x^4} & \\
\qquad \underline{1-\frac{1}{x^2}} & \\
\qquad\quad \frac{1}{x^2}-\frac{1}{x^4} & \\
\qquad\quad \underline{\frac{1}{x^2}-\frac{1}{x^4}} &
\end{array}$$

12. Divide $\frac{1}{1+x}+\frac{x}{1-x}$ by $\frac{1}{1-x}-\frac{x}{1+x}$. *Quot.*, 1.

Suggestion.

$$\frac{1}{1+x}+\frac{x}{1-x}=\frac{1+x^2}{1-x^2} \text{ and } \frac{1}{1-x}-\frac{x}{1+x}=\frac{1+x^2}{1-x^2}.$$

13. Divide $\frac{x+y}{x-y}+\frac{x-y}{x+y}$ by $\frac{x+y}{x-y}-\frac{x-y}{x+y}$.

Quot., $\frac{x^2+y^2}{2xy}$.

14. Divide $\frac{a^6-b^6}{a^2-2ab+b^2}$ by $\frac{a^2+ab+b^2}{a-b}$. *Quot.*, a^3+b^3.

15. Divide $\frac{9x^2-4y^2}{x+y}$ by $\frac{3x-2y}{x^2-y^2}$.

Quot., $3x^2-xy-2y^2$.

Scholium 2.—It is sometimes convenient to write the divisor under the dividend in the form of a complex fraction, and then reduce the result to a simple fraction by (**165**).

16. Divide $1+\frac{1}{a}$ by $1-\frac{1}{a^2}$.

Operation. $$\frac{\left[1+\frac{1}{a}\right]\times a^2}{\left[1-\frac{1}{a^2}\right]\times a^2}=\frac{a^2+a}{a^2-1}=\frac{a}{a-1}.$$

17. Divide $\frac{1}{a}+\frac{1}{ab^3}$ by $b+\frac{1}{b}-1$.

Operation. $$\frac{\left[\frac{1}{a}+\frac{1}{ab^3}\right]\times ab^3}{\left[b+\frac{1}{b}-1\right]\times ab^3}=\frac{b^3+1}{ab^4+ab^2-ab^3}$$

$$=\frac{b^3+1}{ab^2\,(b^2-b+1)}=\frac{b+1}{ab^2}.$$

18. Divide $x^{-3}y-y^2$ by $xy^{-2}+x^{-2}y$.

Suggestions. $$\frac{x^{-3}y-y^2}{xy^{-2}+x^{-2}y}=\frac{\left[\frac{y}{x^3}-y^2\right]\times x^3y^2}{\left[\frac{x}{y^2}+\frac{y}{x^2}\right]\times x^3y^2}=\frac{y^3-x^3y^4}{x^4+xy^3}.$$

19. Free $\frac{x^2-y^{-2}}{ab^{-3}+x^{-1}}$ from negative exponents.

Result, $\frac{b^3x^3y^2-b^3x}{axy^2+b^3y^2}$.

20. Free $\frac{a^{-3}-b^{-2}}{a^{-1}+b^{-3}}$ from negative exponents.

Result, $\frac{b^3-a^3b}{a^2b^3+a^3}$.

21. Free $\frac{x^2+y^{-3}}{mn^{-4}e^{-2}}$ from negative exponents.

Ans., $\frac{e^2n^4x^2y^3+e^2n^4}{my^3}$.

22. Free $\frac{1-a^{-2}-y^2}{1-x^{-3}y^{-2}+x^{-2}}$ from negative exponents.

Ans., $\frac{a^2x^3y^2-x^3y^2-a^2x^3y^4}{a^2x^3y^2-a^2+a^2xy^2}$.

23. What is the reciprocal of $\frac{2xy}{a^2-b^2}$?

Solution.—The reciprocal of a quantity being the quotient of 1 divided by that quantity, the reciprocal of $\frac{2xy}{a^2-b^2}$ is $1\div\frac{2xy}{a^2-b^2}$, or $\frac{a^2-b^2}{2xy}$.

175. **Scholium 3.**—Thus it appears that the reciprocal of a fraction is the fraction inverted.

The reciprocal of a quantity with a negative exponent is the same quantity with a positive exponent.

24. What is the reciprocal of $\frac{x^{-2}y^{-3}}{a^{-\frac{1}{2}}b^{-\frac{1}{3}}}$. *Ans.*, $\frac{x^2y^3}{a^{\frac{1}{2}}b^{\frac{1}{3}}}$.

25. What is the reciprocal of $\frac{(a-b)^{-5}}{(a-b)^{-2}}$?

Ans., $a^3-3a^2b+3ab^2-b^3$.

[To be taken in review.]

26. Multiply $\frac{a^{\frac{2}{3}}-b^{\frac{2}{3}}}{a^{\frac{4}{3}}-b^{\frac{2}{3}}}$ by $\frac{a^{\frac{2}{3}}+b^{\frac{1}{3}}}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}$. *Prod.*, $\frac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{a^{\frac{2}{3}}-b^{\frac{1}{3}}}$.

27. Divide $\frac{x^{-\frac{4}{3}}-x^{-\frac{1}{3}}}{2x^{-3}}$ by $\frac{x^{-\frac{2}{3}}+x^{-\frac{1}{6}}}{4x^2\,(x^{-\frac{2}{3}}-x^{-\frac{1}{6}})}$.

Quot., $2x^{\frac{11}{3}}-4x^{\frac{25}{6}}+2x^{\frac{14}{3}}$.

SYNOPSIS FOR REVIEW.

- **FRACTIONS.**
  - **DEFINITIONS.**
    - **and Fund. Principles.**
      - **FRACTION.**
        - NUMERATOR.
        - DENOMINATOR.
          - Terms.
          - Value of fraction.
          - *Cor.* 1.—Multiplying or dividing numerator or denominator.
          - *Cor.* 2.—Removing denominator.
      - **FORMS.**
        - INTEGRAL.
        - FRACTIONAL.
        - MIXED.
          - Proper.
          - Improper.
            - *Simple.*
            - *Compound.*
            - *Complex.*
      - Lowest terms.—L. C. D.—Reduction.
    - **SIGNS.**
      - Of numerator, of denominator, of fraction.
      - Essential sign of fraction.
  - **OPERATIONS.**
    - **Reductions.**
      - PROB. 1.—To lowest terms. RULE. *Dem.* *Sch.* By H. C. D.
      - PROB. 2.—From improper to integral or mixed forms. RULE. *Dem.* *Cor.* Negative exponents.
      - PROB. 3.—From integral or mixed to fractional forms. RULE. *Dem*
      - PROB. 4.—To common denominators. RULE. *Dem.* *Cor.* L. C. D. *Dem.*
      - PROB. 5.—Complex to simple. RULE. *Dem.*
    - **Fund. Combinations.**
      - **ADDITION.** PROB. RULE. *Dem.* *Cor.* Mixed Numbers.
      - **SUBTRACTION.** PROB. RULE. *Dem.* *Cor.* Mixed Numbers.
      - **MULTIPLICATION.**
        - PROB. 1.—Fraction by integer. RULE. *Dem.* *Sch.*
        - PROB. 2.—Any number by fractions. RULE. *Dem.* *Sch.* *Cor.* Mixed Numbers.
      - **DIVISION.**
        - PROB. 1.—Fractions by integer. RULE. *Dem.*
        - PROB. 2.—Any number by fraction. RULE. *Dems.* 1, 2, 3. *Schs.* 1, 2, 3.

Test Questions.—Upon what five principles in Division are most of the operations in fractions based? Why does the process of reducing to forms having a common denominator not change the value of a fraction? Give the rules for Multiplication and Division of Fractions, and the reasons for them.

Powers and Roots.

CHAPTER IV.

SECTION I.

INVOLUTION.

The operations in Radicals are all based upon the most elementary principles of Factoring.*

If the student learns how to use this key, he can unlock all the mysteries of the subject.

GENERAL DEFINITIONS.

***176.* A Power** is a *product* arising from multiplying a number by itself.

The Degree of the power is indicated by the number of factors taken.

Thus 2, 4, 8, 16, and 32 are, respectively, the 1st, 2nd, 3d, 4th, and 5th powers of 2.

Scholium.—It will be seen that a power is a species of composite number in which the component factors are equal.

***177.* A Root** is one of the equal factors into which a number is conceived to be resolved.

The Degree of the root is indicated by the number of required factors.

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* The subjects treated in this chapter are among the most difficult, if not actually the most difficult for the pupil in the whole science. In the examination of hundreds of students from all parts of the country, the author has found that the rule is that they are *deficient in knowledge of Radicals.* An attempt is here made to assist the teacher in remedying this defect, by constantly holding the attention to this one central principle.

Thus, 2 is the 1st root of 2, the 2nd root of 4, the 3rd root of 8, the 4th root of 16, the 5th root of 32, etc.

178. **Scholium 1.**—*Power* and *Root* are correlative terms. Thus 32 is the 5th power of 2, and 2 is the 5th root of 32.

179. **Scholium 2.**—The *Second Power* is also called the *Square;* the *Third Power*, the *Cube;* and, sometimes, the *Fourth Power*, the *Biquadrate.* In like manner the 2nd root is called the square root; the third root, the cube root; the fourth root, the biquadrate root. These are Geometrical terms which have been transferred to other branches of mathematics. The second power is called the square, because, if a number represents the side of a square, its second power represents its area, or the square itself. Conversely, if a number represents the area of a square, the square root represents the side. Also the third power represents the volume of a cube, the edge of which is the first power or cube root. Biquadrate means twice squared, and hence the fourth power.

180. **An Exponent** or **Index,** is a number written a little to the right and above another number, and indicates

1st. *If a Positive Integer,* a *Power* of the number;

2nd. *If a Positive Fraction,* the numerator indicates a *Power,* and the denominator a *Root* of the number;

3d. *If a Negative Integer or Fraction,* it indicates the *Reciprocal* of what it would signify if positive.

Illustration. 4^3 is the 3rd power of 4, or 64. a^m is the mth power of a, if m is an integer. 4^{-3} (read "4, exponent -3") is $\frac{1}{4^3}$, or $\frac{1}{64}$. a^{-m} is $\frac{1}{a^m}$. $8^{\frac{2}{3}}$ is the cube root of the square, or the square of the cube root of 8, or 4. $a^{\frac{m}{n}}$ is the mth power of the nth root of a, or the nth root of the mth power, if m and n are both integers. $a^{-\frac{m}{n}}$ is $\frac{1}{a^{\frac{m}{n}}}$.

Scholium.—It is obviously incorrect to read $4^{\frac{2}{5}}$, "the $\frac{2}{5}$ *power* of 4." There is no such thing as a 2-fifths power, as will be seen by considering the definition of a power. Read $4^{\frac{2}{5}}$, "4 exponent $\frac{2}{5}$;"

also $a^{\frac{m}{n}}$, "a exponent $\frac{m}{n}$"; $a^{-\frac{m}{n}}$, "a exponent $-\frac{m}{n}$." These are abbreviated forms for, "a with an exponent $-\frac{m}{n}$," etc. In this way any exponent, however complicated, is read without difficulty.

181. **Cor.**—*A* FACTOR *can be transferred from numerator to denominator of a fraction, or* VICE VERSA, *by changing the sign of its exponent, without altering the value of the fraction.*

Thus $\frac{a^m x^{-n}}{4y^{-3}} = \frac{a^m y^3}{4x^n}$; for $\frac{a^m x^{-n}}{4y^{-3}} = \frac{a^m \frac{1}{x^n}}{4\frac{1}{y^3}} = \frac{\frac{a^m}{x^n}}{\frac{4}{y^3}} = \frac{a^m}{x^n} \times \frac{y^3}{4} = \frac{a^m y^3}{4x^n}$.

182. **A Radical Number** is an indicated root of a number. If the root can be extracted exactly, the quantity is called *Rational;* if the root can not be extracted exactly, the expression it called *Irrational,* or *Surd.*

Thus the radical $\sqrt{25a^2}$ is rational, but $\sqrt{13a}$ is surd.

183. A *Root is indicated* either by the denominator of a fractional exponent, or by the **Radical Sign**, $\sqrt{}$. This sign used alone signifies square root. Any other root is indicated by writing its index in the opening of the v part of the sign.

Thus $\sqrt[3]{am}$, $\sqrt[5]{am}$, are the 3rd and 5th roots of am, and the same as $(am)^{\frac{1}{3}}$, $(am)^{\frac{1}{5}}$.

184. **An Imaginary Quantity** is an indicated *even* root of a negative quantity, and is so called because no number taken an *even* number of times as a factor, produces a negative quantity.

Thus $\sqrt{-4}$ is imaginary, because no number multiplied by itself once produces -4. Neither $+2$ nor -2 produces -4 when squared. For a like reason $\sqrt{-3a^2}$, $\sqrt{-5x}$, or $\sqrt[4]{-140xy^2}$ are imaginaries.

185. All quantities not imaginary are called *Real.*

186. Similar Radicals are like roots of like quantities.

Thus $4\sqrt{5a}$, $3x\sqrt{5a}$, and $(a^2-x^2)\sqrt{5a}$ are similar radicals.

187. To Rationalize an expression is to free it from radicals.

188. To affect a number with an Exponent is to perform upon it the operations indicated by that exponent.

Thus to affect 8 with the exponent $\frac{2}{3}$ is to extract the cube root of the square of 8, or to square its cube root, and gives 4.

189. Involution is the process of raising numbers to required powers.

190. Evolution is the process of extracting roots of numbers.

191. Calculus of Radicals treats of the processes of reducing, adding, subtracting, or performing any of the common arithmetical operations upon radical quantities.

INVOLUTION.

192. Prob. 1.—To raise a number to any required power.

Rule.—*Multiply the number by itself as many times, less one, as there are units in the degree of the power.*

Demonstration.—Since the number of factors taken to produce a power, is equal to the degree of the power (**176**), it follows that to obtain the 2nd power we take two factors, or multiply the number by itself *once;* to obtain the 3rd power we take three factors, or multiply the number by itself *twice;* and in like manner to obtain the nth power we take n factors, or multiply the number by itself $n-1$ times.

EXAMPLES.

1. What is the 3rd power of $2a^2$?

Model Solution.—Since the 3rd power of $2a^2$ is the product arising from taking it 3 times as a factor, I have $2a^2 \times 2a^2 \times 2a^2 = 8a^6$.

2. What is the 4th power of $-10a^{\frac{1}{2}}$?

$(-10a^{\frac{1}{2}}) \times (-10a^{\frac{1}{2}}) \times (-10a^{\frac{1}{2}}) \times (-10a^{\frac{1}{2}}) = 10000a^2$, *Answer.*

3 to 6. What is the square of $-5m^2n$? Of $6ab^{-\frac{1}{2}}$? Of $-\frac{3}{5}a^{-n}$? Of $-\frac{3a^m}{5b^2}$?

Answers, $25m^4n^2$, $36\frac{a^2}{b}$, $\frac{9}{25a^{2n}}$, $\frac{9}{25}a^{2m}b^{-4}$.

7 to 10. What is the cube of $-2ab^2$? Of $12m^{-\frac{1}{2}}$? Of $-\frac{2}{3}x^n$? Of $-\frac{m^2}{6n}$?

Answers, $-8a^3b^6$, $1728m^{-\frac{3}{2}} = \frac{1728}{m^{\frac{3}{2}}}$, $-\frac{8}{27}x^{3n}$, $-\frac{m^6}{216n^3}$.

11. What is the square of $2a-3x$? or, $(2a-3x) \times (2a-3x)$?

12. What is the 3rd power of $2a^2+3x$?

Ans., $8a^6+36a^4x+54a^2x^2+27x^3$.

13 to 17. Expand the following: $(1+2x+3x^2)^2$, $(1-x+x^2-x^3)^2$, $(a+b-c)^3$, $(1+2x+x^2)^3$, and $(1-3x+3x^2-x^3)^2$.

Results, $1+4x+10x^2+12x^3+9x^4$, $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$, $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$, and $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

18. Show that $\frac{(27a^4-18a^2b^2-b^4)^2}{64a^2b^4} + \frac{(9a^2-b^2)^3(b^2-a^2)}{64a^2b^4} = b^2$.

19. What is the square of $9x+\frac{1}{x}$? *Ans.,* $81x^2+18+\frac{1}{x^2}$.

20. Expand as above $(4-x^{\frac{1}{2}})^2$. *Result,* $16-8x^{\frac{1}{2}}+x$.

21. Expand $(a^{\frac{1}{2}}+c^{\frac{1}{2}})^4$.

Result, $a^2+4a^{\frac{3}{2}}c^{\frac{1}{2}}+6ac+4a^{\frac{1}{2}}c^{\frac{3}{2}}+c^2$.

193. Cor.—*Since any number of positive factors gives a positive product, all powers of positive monomials are positive. Again, since an* EVEN *number of negative factors gives a* POSITIVE *product, and an* ODD *number gives a* NEGATIVE *product, it follows that even powers of negative numbers are positive, and odd powers negative.*

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194. Prob. 2.—To affect a monomial with any exponent.

Rule.—*Perform upon the coefficient the operations indicated by the exponent, and multiply the exponents of the letters by the given exponent.*

Demonstration.—1st. *When the exponent is a positive integer.*—Let it be required to affect $4a^mb^{\frac{n}{r}}x^{-s}$ with the exponent p; or in other words raise it to the pth power.

It is obvious that, if $4a^mb^{\frac{n}{r}}x^{-s}$ is taken p times, each of its factors will be taken p times.

Now 4 taken p times as a factor is represented thus, 4^p

Again, a^m taken p times as a factor becomes...... a^{mp} because a^m signifies a taken m times as a factor, and if this group of m factors is taken p times, a will have been taken mp times, giving a^{mp}.

In like manner, $b^{\frac{n}{r}}$ signifies $b^{\frac{1}{r}}$ taken as a factor n times, and if this group of n factors is taken p times, $b^{\frac{1}{r}}$ will have taken np times, making.................. $b^{\frac{np}{r}}$

Lastly, x^{-s} signifies $\frac{1}{x^s}$ (**180**, 3d); and this taken as a factor p times give $\frac{1^p}{x^{ps}}$. But as $1^p = 1$, this fraction is by (**181**) x^{-ps}

Collecting the factors we have............... $4^pa^{mp}b^{\frac{np}{r}}x^{-ps}$

Q. E. D.

2nd. *When the exponent is a positive fraction.* Let it be required to affect $4a^mb^{\frac{n}{r}}x^{-s}$, with the exponent $\frac{p}{q}$. This means that $4a^mb^{\frac{n}{r}}x^{-s}$ is to be resolved into q equal factors and p of them taken. Or that we are to find the qth root of the quantity and then raise it to the pth power (**180**, 2d). If we then separate each of the factors of $4a^mb^{\frac{n}{r}}x^{-s}$ into q equal factors, and then take p of each of these, we shall have done what is signified by the exponent $\frac{p}{q}$.

By definition, *one* of the q equal factors of 4 is....... $4^{\frac{1}{q}}$

Also by definition, one of the q equal factors of a^m, or the qth root of a^m is represented (**183**) by............ $a^{\frac{m}{q}}$

To separate $b^{\frac{n}{r}}$ into q equal factors, we notice that $b^{\frac{n}{r}}$ is n of the r equal factors of b. Now, if we resolve each of these r factors into q equal factors, b is resolved into rq equal factors; doing the same with each of the n factors represented by $b^{\frac{n}{r}}$, and taking *one* from each set, we have.. $b^{\frac{n}{rq}}$

which is therefore *one* of the q equal factors of $b^{\frac{n}{r}}$.

To resolve $x^{-s} = \frac{1}{x^s}$ into q equal factors, we consider that a fraction is resolved by resolving its numerator and denominator separately. But one of the q equal factors of 1 is 1; and *one* of the q equal factors of x^s is $x^{\frac{s}{q}}$ as seen in the resolution of a^m. Hence *one* of the q equal factors of x^{-s}, or $\frac{1}{x^s}$, is $\frac{1}{x^{\frac{s}{q}}} =$.................................. $x^{-\frac{s}{q}}$

Collecting these factors we find that *one* of the q equal factors of $4a^mb^{\frac{n}{r}}x^{-s}$ is.............................. $\overline{4^{\frac{1}{q}}a^{\frac{m}{q}}b^{\frac{n}{rq}}x^{-\frac{s}{q}}}$

And finally p of these being obtained according to Case 1st, gives $4^{\frac{p}{q}}a^{\frac{pm}{q}}b^{\frac{pn}{rq}}x^{-\frac{ps}{q}}$, as the expression for $4a^mb^{\frac{n}{r}}x^{-s}$ affected with the exponent $\frac{p}{q}$; which result agrees with the enunciation of the rule.

3rd. *When the exponent is negative and either integral or fractional.* Let it be required to affect $4a^mb^{\frac{n}{r}}x^{-s}$ with the exponent $-t$. This, by definition of negative exponents, signifies that we are to take the reciprocal of what the expression would be if t were positive. But $4a^mb^{\frac{n}{r}}x^{-s}$ affected with the exponent t (positive) is $4^ta^{tm}b^{\frac{tn}{r}}x^{-ts}$, whether t is integral or fractional, as shown in the preceding cases. The reciprocal of this is $\dfrac{1}{4^ta^{tm}b^{\frac{tn}{r}}x^{-ts}}$. But since these factors can be transferred to the numerator by changing the signs of their exponents, we have $4^{-t}a^{-tm}b^{-\frac{tn}{r}}x^{ts}$, as the result of affecting $4a^mb^{\frac{n}{r}}x^{-s}$ with the exponent $-t$, which result agrees with the enunciation of the rule.

[**Note.**—The above demonstration contains the fundamental principles of the whole subject of the Theory of Exponents, and it is of the highest importance that it be made perfectly familiar. The application of the rule is so simple in practice as to afford no difficulty; but the reasoning should be given in full in a sufficient number of the following examples *to fix in the mind these principles of the theory.* After this is done, the pupil needs only to perform the operations. The danger is that the *how* being so simple, the *why* will be disregarded.]

EXAMPLES.

Ex. 1. Affect $2a^2b^{\frac{2}{3}}c^{-4}$ with the exponent 5: that is, raise it to the 5th power.

Model Solution.

Operation. $(2a^2b^{\frac{2}{3}}c^{-4})^5 = 32a^{10}b^{\frac{10}{3}}c^{-20}$.

Explanation. $(2a^2b^{\frac{2}{3}}c^{-4})^5$ is $2a^2b^{\frac{2}{3}}c^{-4} \times 2a^2b^{\frac{2}{3}}c^{-4} \times 2a^2b^{\frac{2}{3}}c^{-4}$ - - - - to 5 factors. This gives 5 factors of 2, or 32; 5 factors of a^2 or 10 factors of a, a^{10}; 5 factors of $b^{\frac{2}{3}}$, which I obtain by considering that $b^{\frac{2}{3}}$ is 2 factors of $b^{\frac{1}{3}}$, and hence 5 factors of $b^{\frac{2}{3}}$ is 5 times 2 factors of $b^{\frac{1}{3}}$, or $b^{\frac{10}{3}}$; and since $c^{-4} = \frac{1}{c^4}$, 5 factors of it are $\frac{1}{c^4} \times \frac{1}{c^4}$ - - to

5 factors $= \frac{1}{c^{20}} = c^{-20}$, as fractions are multiplied by multiplying numerators and denominators. Hence I have $(2a^2b^{\frac{2}{3}}c^{-4})^5 = 32a^{10}b^{\frac{10}{3}}c^{-20}$.

2. Affect $3a^{\frac{2}{3}}$ with the exponent m; that is, raise it to the mth power, m being an integer.

Explanation. $(3a^{\frac{2}{3}})^m = 3a^{\frac{2}{3}} \times 3a^{\frac{2}{3}} \times 3a^{\frac{2}{3}}$ - - - to m factors. This gives m factors of 3 and m factors of $a^{\frac{2}{3}}$. But m factors of 3 are represented by 3^m. To obtain m factors of $a^{\frac{2}{3}}$, I consider that $a^{\frac{2}{3}}$ is 2 factors of $a^{\frac{1}{3}}$; hence m factors of it are $2m$ factors of $a^{\frac{1}{3}}$. That is, a resolved into 3 equal factors and $2m$ of them taken. Therefore $(3a^{\frac{2}{3}})^m = 3^m a^{\frac{2m}{3}}$.

3. Affect $16a^{15}x^{-6}$ with the exponent $\frac{2}{3}$; that is, represent 2 of the 3 equal factors of $16a^{15}x^{-6}$.

Explanation.—I will first resolve $16a^{15}x^{-6}$ into 3 equal factors (or indicate it when I cannot perform it). To do this I take one of the 3 equal factors of 16, which I *represent*, as I cannot resolve it, and write $(16)^{\frac{1}{3}}$. Again, one of the 3 equal factors of a^{15} is a^5, as a^{15} is 15 factors of a, and consequently when resolved into 3 equal factors one of the 3 contains 5 factors of a. Thus $a \times a \times a$ - - - - to 15 factors when put into 3 equal groups becomes $aaaaa \times aaaaa \times aaaaa$, one of which is a^5. To resolve x^{-6} I consider it as $\frac{1}{x^6} = \frac{1}{x} \times \frac{1}{x} \times \frac{1}{x} \times \frac{1}{x} \times \frac{1}{x} \times \frac{1}{x}$. Combining this into 3 equal groups it becomes $\frac{1}{x^2} \times \frac{1}{x^2} \times \frac{1}{x^2}$; hence one of the 3 equal factors of $\frac{1}{x^6}$ is $\frac{1}{x^2}$, or x^{-2}. Therefore $16a^{15}x^{-6}$ being resolved into 3 equal factors, one of them is $(16)^{\frac{1}{3}}a^5x^{-2}$. But I am to take 2 of these, as the exponent $\frac{2}{3}$ indicates. This gives me (repeat the reasoning of Ex. 1) $(16)^{\frac{2}{3}}a^{10}c^{-4}$.

4. Affect $3a^nb^{-m}$ with the exponent $\frac{m}{n}$; that is, take m of the n equal factors of it.

Explanation. $3^{\frac{1}{n}}$ *represents* one of the n equal factors of 3. One of the n equal factors of a^n is a, as a^n means n factors of a. $b^{-m} = \frac{1}{b^m} = \frac{1}{b} \times \frac{1}{b} \times \frac{1}{b} \times$ - - - - to m factors. Each of these being separated into n equal factors, I have $\frac{1}{b^{\frac{1}{n}}} \times \frac{1}{b^{\frac{1}{n}}} \times \frac{1}{b^{\frac{1}{n}}}$ - - - - to n factors $= \frac{1}{b}$. And taking *one* of the factors $\frac{1}{b^{\frac{1}{n}}}$, from each of the m factors $\frac{1}{b}$, I have m factors of $\frac{1}{b^{\frac{1}{n}}}$ or $\frac{1}{b^{\frac{m}{n}}} = b^{-\frac{m}{n}}$. Therefore *one* of the n equal factors of $3a^nb^{-m}$ is $3^{\frac{1}{n}}ab^{-\frac{m}{n}}$. But I am to take m such factors which gives (repeat the process of Ex. 2) $3^{\frac{m}{n}}a^mb^{-\frac{m^2}{n}}$.

5. What is the square of $\frac{2}{3}a^3m$? Of $-\frac{3ab}{2x^3}$, or $-\frac{3}{2}abx^{-3}$?

The cube of $-\frac{2a^{\frac{1}{2}}x}{3m^{\frac{1}{3}}}$, or $-\frac{2}{3}a^{\frac{1}{2}}m^{-\frac{1}{3}}x$?

6. Affect $8a^{12}x^{-6}$ with the exponent $\frac{2}{3}$. *Result*, $4a^8x^{-4}$.

7. Affect $32a^{20}y^{15}$ with the exponent $-\frac{4}{5}$.

Result, $\frac{1}{16a^{16}y^{12}}$.

8. Affect $13x^{-\frac{1}{3}}y^{-3}$ with the exponent -3.

Result, $\frac{1}{2197}xy^9$.

9. Perform the following operations and explain each as a process of factoring: $(32a^{\frac{20}{3}}b^{-15})^{-\frac{3}{5}}$, $(100a^{-2}b^{\frac{2}{3}})^{\frac{1}{2}}$, $(11a^2x^{-\frac{1}{2}}y^{-2})^{-\frac{1}{2}}$, and $(a^mb^{\frac{m}{n}}x^{-s})^{-\frac{r}{t}}$.

Results, $\frac{b^9}{8a^4}$, $10\frac{b^{\frac{1}{3}}}{a}$, $\frac{x^{\frac{1}{4}}y}{(11)^{\frac{1}{2}}a}$, and $\frac{x^{\frac{rs}{t}}}{a^{\frac{mr}{t}}b^{\frac{mr}{nt}}}$.

***195.* Prob. 3.**—To expand a binomial affected with any exponent.

Rule.—*This rule is best stated in a formula. Thus, let* a, b, *and* m *be any numbers whatever, positive or negative, integral or fractional, then will* $(a+b)^m$ *represent any binomial, affected with any exponent, and*

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1 \cdot 2}a^{m-2}b^2$$

$$+ \frac{m(m-1)\ (m-2)}{1 \cdot 2 \cdot 3}a^{m-3}b^3$$

$$+ \frac{m(m-1)\ (m-2)\ m-3)}{1 \cdot 2 \cdot 3 \cdot 4}a^{m-4}b^4$$

$$+ \frac{m(m-1)\ (m-2)\ (m-3)\ (m-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}a^{m-5}b^5 + \text{etc.}$$

Demonstration.—This formula is the celebrated BINOMIAL THEOREM discovered by Sir Isaac Newton. There are several elementary demonstrations, one of which will be found in Appendix I. If the pupil learns the formula, and learns to apply it with facility, it is all that is thought best for him to attempt at this stage of his progress.

EXAMPLES.

Ex. 1. Expand $(x+y)^5$ by the Binomial Formula.

Model Solution.—To apply the B. F., $x = a$ of the formula, $y = b$, and $5 = m$. $\therefore$ I have $(x+y)^5 = x^5 + 5x^{5-1}y + \frac{5\,(5-1)}{2}$ $x^{5-2}y^2 + \frac{5\,(5-1)\,(5-2)}{2\ \cdot\ 3}\,x^{5-3}y^3 + \frac{5\,(5-1)\,(5-2)\,(5-3)}{2\ \cdot\ 3\ \cdot\ 4}\,x^{5-4}y^4$ $+ \frac{5\,(5-1)\,(5-2)\,(5-3)\,(5-4)}{2\ \cdot\ 3\ \cdot\ 4\ \cdot\ 5}\,x^{5-5}y^5$, at which term the development becomes complete since the next coefficient would have a factor $5-5=0$ which would destroy the term. Performing the operations indicated I have $(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$. (In practice, this result should be written out without writing the preceding, by simply applying the formula mentally.)

2. Expand $(x-y)^6$ by the B. F.

Suggestions. $x=a$, $-y=b$, and $6=m$. $\therefore$ $(x-y)^6 = x^6 + 6x^5(-y) + \frac{6\cdot 5}{2}x^4(-y)^2 + \frac{6\cdot 5\cdot 4}{\underline{|3}^{*}}x^3(-y)^3 + \frac{6\cdot 5\cdot 4\cdot 3}{\underline{|4}}x^2(-y)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{\underline{|5}}x(-y)^5 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{\underline{|6}}x^0(-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.

3. Expand $(2m^2-3n^{\frac{1}{2}})^4$ by the B. F.

Suggestions.—Make $a=2m^2$, $b=-3n^{\frac{1}{2}}$, and $m=4$, and $(2m^2-3n^{\frac{1}{2}})^4 = 16m^8-96m^6n^{\frac{1}{2}}+216m^4n-216m^2n^{\frac{3}{2}}+81n^2$.

4. Expand $(x+y)^{-4}$ by the B. F.

Suggestions. $a=x$, $b=y$, and $m=-4$. This series does not terminate, since no factor ever becomes 0; but the development can be carried to any desired extent. $(x+y)^{-4} = x^{-4}-4x^{-5}y+10x^{-6}y^2-20x^{-7}y^3+35x^{-8}y^4-$, etc.

5. Expand $(m-n)^{\frac{2}{3}}$ by the B. F.

Suggestions.—Making the proper substitutions in the formula, we have $(m-n)^{\frac{2}{3}} = m^{\frac{2}{3}} + \frac{2}{3}m^{\frac{2}{3}-1}(-n) + \frac{\frac{2}{3}(\frac{2}{3}-1)}{2}m^{\frac{2}{3}-2}(-n)^2 + \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{\underline{|3}}m^{\frac{2}{3}-3}(-n)^3 + \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)(\frac{2}{3}-3)}{\underline{|4}}m^{\frac{2}{3}-4}(-n)^4 +$, etc. A series which never terminates, since no coefficient reduces to 0. Performing the operations indicated we find that $(m-n)^{\frac{2}{3}} = m^{\frac{2}{3}} - \frac{2}{3}m^{-\frac{1}{3}}n - \frac{1}{9}m^{-\frac{4}{3}}n^2 - \frac{4}{81}m^{-\frac{7}{3}}n^3 - \frac{7}{243}m^{-\frac{10}{3}}n^4 -$, etc.

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[*] $\underline{|3}$ is read "factorial 3," and means the product of the natural numbers from 1 to 3 inclusive. $\underline{|5}$ is the same as $1\times 2\times 3\times 4\times 5$, etc.

6. Expand $(1+x)^n$ by the B. F.

Result, $(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)\ (n-2)}{\underline{|3}}x^3$ $+\frac{n(n-1)\ (n-2)\ (n-3)}{\underline{|4}}x^4+$, etc. $--\frac{n(n-1)}{2}x^{n-2}+nx^{n-1}$ $+x^n$.

Scholium.—This expansion is in itself a very useful formula, and should be memorized.

7. Expand $(3-y^2)^{\frac{1}{2}}$ by the B. F.

Result, $(3-y^2)^{\frac{1}{2}}=3^{\frac{1}{2}}-\frac{3^{-\frac{1}{2}}}{2}y^2-\frac{3^{-\frac{3}{2}}}{8}y^4-\frac{3^{-\frac{5}{2}}}{16}y^6$ $-\frac{5\cdot 3^{-\frac{7}{2}}}{128}y^8-$, etc.

8. Expand $(1+x^2)^5$ by the B. F.

Result, $1+5x^2+10x^4+10x^6+5x^8+x^{10}$.

9. Expand $(1-a^2)^{-\frac{1}{2}}$ by the B. F.

Result, $1+\frac{1}{2}a^2+\frac{3}{8}a^4+\frac{5}{16}a^6+\frac{35}{128}a^8+$, etc.

10. Expand $\sqrt{a^2-a^2e^2}$ by the B. F.

Result, $\sqrt{a^2-a^2e^2}=a\sqrt{1-e^2}=a(1-e^2)^{\frac{1}{2}}=a(1-\frac{1}{2}e^2$ $-\frac{1}{2\cdot 4}e^4-\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}e^8-$, etc.).

11. Expand $\frac{1}{(c+x)^2}$ by the B. F.

Result, $\frac{1}{(c+x)^2}=(c+x)^{-2}=c^{-2}-2c^{-3}x+3c^{-4}x^2-4c^{-5}x^3$ $+$ etc. $=\frac{1}{c^2}(1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+$, etc.).

12. Expand $\frac{1}{(1+x)^{\frac{1}{5}}}$ by the B. F.

Result, $\frac{1}{(1+x)^{\frac{1}{5}}} = (1+x)^{-\frac{1}{5}} = 1 - \frac{x}{5} + \frac{3x^2}{25} - \frac{11x^3}{125} + \frac{44x^4}{625} -$, etc.

13. Expand $(x+y+c)^4$ by the B. F.

Suggestions.—Put $(x+y)=z$. $\therefore (x+y+c)^4 = (z+c)^4 = z^4 + 4z^3c + 6z^2c^2 + 4zc^3 + c^4 =$ (restoring the value of z) $(x+y)^4 + 4(x+y)^3c + 6(x+y)^2c^2 + 4(x+y)c^3 + c^4$. But $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, and $(x+y)^2 = x^2 + 2xy + y^2$. Whence by substitution we have $(x+y+c)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 + 4cx^3 + 12cx^2y + 12cxy^2 + 4cy^3 + 6c^2x^2 + 12c^2xy + 6c^2y^2 + 4c^3x + 4c^3y + c^4$.

14. Expand $(2a-b+c^2)^3$ by the B. F.

Result, $8a^3 - 12a^2b + 6ab^2 - b^3 + 12a^2c^2 - 12abc^2 + 3b^2c^2 + 6ac^4 - 3bc^4 + c^6$.

196. **Cor. 1.**—*The expansion of a binomial terminates only when the exponent is a positive integer, since only when* m *is a positive integer will a factor of the form* m(m−1)(m−2)(m−3) *etc. become* 0.

197. **Cor. 2.**—*When* m *is a positive integer, that is when a binomial is raised to any power, there is one more term in the development than units in the exponent.* Since the first coefficient is 1; the 2nd, m; the 3rd, $\frac{m(m-1)}{2}$; the 4th, $\frac{m(m-1)(m-2)}{\underline{|3}}$; the 5th, $\frac{m(m-1)(m-2)(m-3)}{\underline{|4}}$; etc., we notice that the last factor is $m-$ (the number of the term -2); and the number of the term, therefore, which has $m-m$ as a factor is the $(m+2)$th term. But this is 0. Hence the $(m+1)$th term is the last.

198. Cor. 3.—*When* m *is a positive integer the coefficients equally distant from the extremes are equal;* since $(a+b)^m = (b+a)^m$; the former of which gives $a^m + ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 +$, etc., and the latter $b^m + mb^{m-1}a + \frac{m(m-1)}{2}b^{m-2}a^2 +$, etc. Whence it appears that the first half of the terms and the last half are exactly symmetrical.

199. Cor. 4.—*The sum of the exponents in each term is the same as the exponent of the power.*

Scholium.—The last two corollaries apply to the form $(x+y)^m$, and not to such forms as $(2a^3 - 3b^2)^m$, after the latter is fully expanded.

200. Cor. 5.—*A convenient rule for writing out the* POWERS *of binomials may be thus stated:*

1. *The* FIRST *term contains only the first letter of the binomial, and the last term only the second, while all the other terms contain both the letters.*

2. *The exponent of the first letter of the binomial in the first term of the development is the same as the exponent of the required power and* DIMINISHES *by unity to the right, while the exponent of the second letter begins at unity in the second term of the expansion and* INCREASES *by unity to the right, becoming, in the last term, the same as the exponent of the power.*

3. *The coefficient of the first term of the expansion is unity; of the second, the exponent of the required power; and that of any other term may be found by multiplying the coefficient of the preceding term by the exponent of the first letter in that term, and dividing the product by the exponent of the second letter* $+1$.

4. *There is one more term in the development than there are units in the exponent of the power.*

This rule is a deduction from the formula $(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{\underline{|3}}a^{m-3}b^3 + \frac{m(m-1)(m-2)(m-3)}{\underline{|4}}a^{m-4}b^4$ - - - - etc.

The 1st point appears from Cor. 3.

The law of the exponents is directly observable from the formula.

The coefficients of the first and second terms are seen in the formula to be as stated. The coefficient of the *third* term may be written $m \times \frac{m-1}{2}$, which is the coefficient of the *second*, or preceding term (m), multiplied by the exponent ($m-1$) of the first letter in that term, and divided by 2 which is the exponent (1) of the second letter + 1. In like manner noticing any other term, as the 5th, its coefficient may be written $\frac{m(m-1)(m-2)}{\underline{|3}} \times \frac{m-3}{4}$. But $\frac{m(m-1)(m-2)}{\underline{|3}}$ is the coefficient of the 4th term, $m-3$ is the exponent of the first letter in the 4th term, and 4, the divisor, is the exponent of the second letter (3) + 1.

The 4th point appears from Cor. 2.

15 to 20. Write out by the above rule the expansions of the following: $(m+n)^4$, $(x+y)^5$, $(a+c)^7$, $(x+m)^8$, $(m^2+a^3)^3$, $(a^{\frac{2}{3}}+m^{\frac{1}{2}})^4$.

Suggestions upon the last.—Regard $a^{\frac{2}{3}}$ and $m^{\frac{1}{2}}$ as simple numbers represented by letters without exponents. Thus $(a^{\frac{2}{3}}+m^{\frac{1}{2}})^4 = (a^{\frac{2}{3}})^4 + 4(a^{\frac{2}{3}})^3(m^{\frac{1}{2}}) + 6(a^{\frac{2}{3}})^2(m^{\frac{1}{2}})^2 + 4(a^{\frac{2}{3}})(m^{\frac{1}{2}})^3 + (m^{\frac{1}{2}})^4$. Now performing the operations indicated, we have $(a^{\frac{2}{3}}+m^{\frac{1}{2}})^4 = a^{\frac{8}{3}} + 4a^{\frac{6}{3}}m^{\frac{1}{2}} + 6a^{\frac{4}{3}}m + 4a^{\frac{2}{3}}m^{\frac{3}{2}} + m^2$.

201. Cor. 6.—*If the sign between the terms of the binomial is minus and the exponent is a positive integer, as* $(a-b)^m$, *the odd terms of the expansion are + and the even ones —. This arises from the fact that the odd terms involve even powers of the second or negative term of the binomial, and the even terms involve the odd powers of the same.*

Thus the second term involves $(-b)$ which makes its sign $-$; the 4th term has $(-b)^3$, the 6th term $(-b)^5$, etc. But the first term does not involve $(-b)$, and the 3rd has $(-b)^2$ or b^2, the 5th has $(-b)^4$ or b^4, etc.

21 to **24**. Write out the expansions of $(m-n)^5$, $(x-y)^7$, $(x^2-a^3)^3$, $(a^{\frac{1}{3}}-b^{\frac{1}{3}})^4$.

Result of the last, $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^4 = a^2-4a^{\frac{3}{2}}b^{\frac{1}{3}}+6ab^{\frac{2}{3}}-4a^{\frac{1}{2}}b+b^{\frac{4}{3}}$.

SECTION II.

EVOLUTION.

***202.* Prob. 1.—To extract the *m*th (any root) of a perfect power of that degree.**

Rule.—*Resolve the number into its prime factors, and separate these into* m *equal groups; one of these groups is the root sought.*

Demonstration.—Since the *m*th root (*i. e.*, any root) of a number is one of the *m* equal factors of that number, if a number is resolved into *m* equal factors, as the rule directs, one of them is the *m*th root.

EXAMPLES

Ex. **1**. Extract the cube root of 74088.

Model Solution.—Resolving 74088 into its prime factors I find them to be $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \; 7 \cdot 7 \cdot 7$. These arranged in 3 equal groups give $2 \cdot 3 \cdot 7 \times 2 \cdot 3 \cdot 7 \times 2 \cdot 3 \cdot 7$. Hence $2 \cdot 3 \cdot 7 = 42$ is the cube root of 74088, since it is one of the 3 equal factors.

2 to **5**. Extract in this manner the following: $\sqrt{492804}$, $\sqrt[3]{592704}$, $\sqrt[5]{248832}$, $\sqrt[3]{456533}$. *Roots*, 702, 84, and 12.

6. Extract the square root of $81a^4x^{-2}y^{\frac{2}{3}}z^{-\frac{1}{5}}$.

Model Solution.—The two equal factors of 81 are $9 \cdot 9$; of a^4 $a^2 \cdot a^2$; of x^{-2}, $x^{-1} \cdot x^{-1}$; of $y^{\frac{2}{3}}$, $y^{\frac{1}{3}} \cdot y^{\frac{1}{3}}$; of $z^{-\frac{1}{5}}$, $z^{-\frac{1}{10}} \cdot z^{-\frac{1}{10}}$. Hence

$81a^4x^{-2}y^{\frac{2}{3}}z^{-\frac{1}{5}} = 9a^2x^{-1}y^{\frac{1}{3}}z^{-\frac{1}{10}} \times 9a^2x^{-1}y^{\frac{1}{3}}z^{-\frac{1}{10}}$, and consequently, $9a^2x^{-1}y^{\frac{1}{3}}z^{-\frac{1}{10}}$ is its square root.

7 to 11. Extract $\sqrt{25a^4b^3}$, $\sqrt{64a^{-6}x^{\frac{2}{3}}}$, $\sqrt{49xy^{\frac{1}{3}}}$, $\sqrt{144a^4m^6}$, $\sqrt{\frac{49a^4b^2}{36m^2n}}$. *Roots*, $\pm 5a^2b^{\frac{3}{2}}$, $\pm 8a^{-3}x^{\frac{1}{3}}$, $\pm 7x^{\frac{1}{2}}y^{\frac{1}{6}}$, $\pm 12a^2m^3$, $\pm\frac{7a^2b}{6mn^{\frac{1}{2}}}$.

12 to 15. Extract $\sqrt[3]{125m^6x^{12}}$, $\sqrt[3]{1728x^2y^{\frac{3}{5}}}$, $\sqrt[5]{-32a^{10}y^{-5}}$, $\sqrt[4]{16n^{-\frac{1}{2}}y^8}$.

Roots, $5m^2x^4$, $12x^{\frac{2}{3}}y^{\frac{1}{5}}$, $-2a^2y^{-1}$, and $\pm 2n^{-\frac{1}{8}}y^2$.

Query.—Why the ambiguous sign to the last?

203. **Scholium.**—The sign of an even root of a positive number is ambiguous (that is, + or −) since an even number of factors gives the same product whether they are positive or negative (**87, 88**). The sign of an odd root is the same as that of the number itself, since an odd number of positive factors gives a positive product and an odd number of negative factors gives a negative product (**88, 89**).

204. **Cor. 1.**—*The roots of monomials can be extracted by extracting the required root of the coefficient and dividing the exponent of each letter by the index of the root, since to extract the square root is to affect a number with the exponent* $\frac{1}{2}$, *the cube root* $\frac{1}{3}$, *the* n*th root* $\frac{1}{n}$, *etc.* (**194.**)

EXAMPLES.

16 to 21. In this manner write $\sqrt{25a^{\frac{1}{3}}b^2}$, $\sqrt[3]{-343x^4y^{-6}}$; $\sqrt[4]{81m^8n^{-\frac{1}{2}}x}$, $\sqrt[5]{243c^{10}m^{-1}y}$, $\sqrt[3]{\frac{27a^6x^5}{64m^3y^6}}$, and $\sqrt[5]{\frac{32a^{15}x}{243b^2y^{-10}}}$.

Roots, $\pm 5a^{\frac{1}{6}}b$, $-7x^{\frac{4}{3}}y^{-2}$, $\pm 3m^2n^{-\frac{1}{8}}x^{\frac{1}{4}}$, $3c^2m^{-\frac{1}{5}}y^{\frac{1}{5}}$, $\frac{3a^2x^{\frac{5}{3}}}{4my^2}$, and $-\frac{2a^3x^{\frac{1}{5}}}{3b^{\frac{2}{5}}y^{-2}}$

205. Cor. 2.—*The root of the product of several numbers is the same as the product of the roots.* Thus, $\sqrt[m]{abcx} = \sqrt[m]{a} \cdot \sqrt[m]{b} \cdot \sqrt[m]{c} \cdot \sqrt[m]{x}$, since to extract the mth root of $abcx$ we have but to divide the exponent of each letter by m, which gives $a^{\frac{1}{m}}b^{\frac{1}{m}}c^{\frac{1}{m}}x^{\frac{1}{m}}$ or $\sqrt[m]{a} \cdot \sqrt[m]{b} \cdot \sqrt[m]{c} \cdot \sqrt[m]{x}$.

206. Cor. 3.—*The root of the quotient of two numbers is the same as the quotient of the roots.* Thus, $\sqrt[r]{\frac{m}{n}}$ is the same as $\frac{\sqrt[r]{m}}{\sqrt[r]{n}}$, since to extract the rth root of $\frac{m}{n}$ we have but to extract the rth root of numerator and denominator, which operation is performed by dividing their exponents by r. Hence $\sqrt[r]{\frac{m}{n}} = \frac{m^{\frac{1}{r}}}{n^{\frac{1}{r}}} = \frac{\sqrt[r]{m}}{\sqrt[r]{n}}$.

EXAMPLES.

22. Show that $\sqrt[3]{8 \times 27} = \sqrt[3]{8} \times \sqrt[3]{27}$.

Model Solution.—We may show this in two ways. 1st.—*Experimentally.* Thus $\sqrt[3]{8 \times 27} = \sqrt[3]{216} = 6$. Again $\sqrt[3]{8} \times \sqrt[3]{27} = 2 \times 3 = 6$. Hence $\sqrt[3]{8 \times 27}$ (or the cube root of the product) $= \sqrt[3]{8} \times \sqrt[3]{27}$ (or the product of the cube roots). 2nd.—*Analytically.* $\sqrt[3]{8 \times 27}$ signifies that the product of 8 and 27 is to be resolved into 3 equal factors, which is accomplished by resolving 8 into its prime factors, and 27 into its prime factors, and then separating these factors into three equal groups (**202**). This will give the same result as resolving the product of 8 and 27, or 216, into 3 equal factors, since the prime factors of 216 are the same as those of 8 and 27.

23. Show that $\sqrt[5]{a^{-m}b^{\frac{1}{n}}} = \sqrt[5]{a^{-m}} \times \sqrt[5]{b^{\frac{1}{n}}}$.

Suggestions.—The 5 equal factors of $a^{-m}b^{\frac{1}{n}}$ are $a^{-\frac{m}{5}}b^{\frac{1}{5n}}$, $a^{-\frac{m}{5}}b^{\frac{1}{5n}}$, $a^{-\frac{m}{5}}b^{\frac{1}{5n}}$, $a^{-\frac{m}{5}}b^{\frac{1}{5n}}$, and $a^{-\frac{m}{5}}b^{\frac{1}{5n}}$, since by the rules of multiplication these multiplied together make $a^{-m}b^{\frac{1}{n}}$. But $a^{-\frac{m}{5}}b^{\frac{1}{5n}}$ is the product of one of the 5 equal factors of a^{-m} by one of the 5 equal factors of $b^{\frac{1}{n}}$, or $\sqrt[5]{a^{-m}}$ by $\sqrt[5]{b^{\frac{1}{n}}}$.

24. Extract the square root of $a^4c^2+2a^3bc^2+a^2b^2c^2$.

Solution.—The factors of this are readily seen to be a^2, c^2, and $a^2+2ab+b^2$, which separated into two equal groups give $ac(a+b)$ and $ac(a+b)$. Hence $ac(a+b)$ or a^2c+abc is the required root.

25. Extract the square root of $m^4-2m^4x+m^4x^2$.

Root, $m^2(1-x)$ or m^2-m^2x.

Scholium.—The extraction of roots by resolving numbers into their factors according to this rule, is limited in its application for several reasons. In the case of decimal numbers we can always find the prime factors by trial, and hence if the number is an exact power, can get its root. But in case the number is not an exact power of the degree required, we have no method of approximating to its exact root by this rule, as we have by the common method already learned in arithmetic. In case of literal numbers the difficulty of detecting the polynomial factors of a polynomial is usually insuperable. Hence we seek general rules which will not be subject to these objections.

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207. Prob. 2.—To extract the square root of a polynomial.

Rule.—*I. Arrange the polynomial with reference to one of its letters, as for division.*

II. Extract the square root of the first left hand term. This root is the first term of the required root. Subtract the square of this term of the root from the polynomial.

III. Double the root already found for a Trial Divisor. By this trial divisor divide the first term of the remainder of the polynomial, and write the quotient as the second term of the root.

IV. Complete the divisor by adding to the trial divisor the last term in the root. Multiply the True Divisor thus formed by the last term in the root, and subtract the product from the last remainder, bringing down such terms as may be necessary.

V. Repeat the process of dividing, completing the divisor, multiplying and subtracting, in the same way till the polynomial is exhausted, or until there is no term of it remaining which can be exactly divided by the first term of the trial divisor.

Demonstration.—1st. The polynomial is arranged as in division, since such is the order which the terms assume in squaring any polynomial root.

2nd. In squaring any polynomial, the first term of the square is found to be the square of the first term in the root; hence, in extracting the square root, the square root of the first term in the given polynomial is the first term in the root.

3rd. To prove the process of finding the divisors and the subsequent terms of the root, we observe the following operations:

$$\text{A.}\quad (a+b)^2 = a^2+2ab+b^2 = \overset{(1)}{a^2}+\overset{(2)}{(2a+b)b}.$$

$$\begin{aligned}\text{B.}\quad (a+b+c)^2 &= [(a+b)+c]^2 = (a+b)^2+[2(a+b)+c]c\\ &= \overset{(1)}{a^2}+\overset{(2)}{(2a+b)b}+\overset{(3)}{[2(a+b)+c]c}.\ *\end{aligned}$$

$$\begin{aligned}\text{C.}\quad (a+b+c+d)^2 &= [(a+b+c)+d]^2 = (a+b+c)^2\\ &+ [2(a+b+c)+d]d = \overset{(1)}{a^2}+\overset{(2)}{(2a+b)b}\\ &+ \overset{(3)}{[2(a+b)+c]c}+\overset{(4)}{[2(a+b+c)+d]d}.\end{aligned}$$

Hence it appears; 1st, That the square of a polynomial (as $a+b$, $a+b+c$, or $a+b+c+d$) is made up of as many parts as there are terms in the root; 2nd, That the first is the square of the first term in the root; 3rd, That the second part is *Twice* the first term of the root (the part already found), + the second term of the root, multiplied by the second term; 4th, That any one of these parts, as the nth, is *Twice* that portion of the root previously found, or *Twice* the $n-1$ preceding terms of the root, + the nth term of the root, multiplied by this last or nth term.

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* This expansion is made by treating $(a+b)+c$ as a binomial, giving as its square $(a+b)^2+2(a+b)c+c^2$. Expanding and factoring we have the form in B.

Scholium 1.—If the first term of the arranged polynomial is not a perfect square, the root can not be extracted.

Scholium 2.—If at any time no term of the remainder can be exactly divided by the first term of the trial divisor, the root can not be extracted.

EXAMPLES.

Ex. 1. Extract the square root of $49a^2x^2-30ax^3+25x^4+16a^4-24a^3x$.

Model Solution.

OPERATION.

$$\begin{array}{rl}
 & 16a^4-24a^3x+49a^2x^2-30ax^3+25x^4 \;\big|\; \underline{4a^2-3ax+5x^2} \\
 & \underline{16a^4} \\
\underline{8a^2-3ax} & \big|\; -24a^3x+49a^2x^2 \\
 & \underline{-24a^3x+\;\;9a^2x^2} \\
\underline{8a^2-6ax+5x^2} & \big|\; 40a^2x^2-30ax^3+25x^4 \\
 & \underline{40a^2x^2-30ax^3+25x^4}
\end{array}$$

Explanation.—If this polynomial is a perfect square, the term containing the highest power of a, or of x, is the square of the first term in the root. Hence I place $16a^4$ first in the arrangement. ($25x^4$ would do as well.) And, as the terms arising from squaring a polynomial (*e. g.* the root of this given number), are arranged according to the leading letter of the root, I arrange the whole polynomial according to the powers of a, as this will be the leading letter of the root, when I put $16a^4$ first in the power.

Having arranged the given number, I know that the square root of the first term, $16a^4$, or $4a^2$ is the first term of the root, since in squaring any polynomial (*e. g.*, the root sought), the first term in the square is the square of the first term in the root.

Now, having removed by subtraction the square of the first term of the root, I double the root already found, obtaining $8a^2$ for a *trial* divisor. This is the trial divisor, since the second part of any square (the square of the first term of the root being called the first part) is twice the root already found + the next term of the root, multiplied by this next term. I now find that the trial divisor is contained in the first term of the remainder $-3ax$ times, which is, therefore, the second term of the root. But the *true* divisor is twice the root previously found plus *this last term;* hence I add $-3ax$

to the trial divisor and have $8a^2-3ax$ as the *true* divisor. Multiplying this true divisor by the last term of the root I obtain $-24a^3x+9a^2x^2$, which is the second part of the given power.

Subtracting this second part of the power, the third part is *twice the root* already *found* + the *next* term of the root, multiplied by this *next* term. In this case, therefore, the new trial divisor is $8a^2-6ax$. [Proceed just as in the last paragraph, and complete the explanation.]

Finally $4a^2-3ax+5x^2$ is the required root, for, indeed, I have actually squared it and subtracted this square from the given number and found no remainder. [The pupil should notice that the sum of the several subtrahends is the square of the root.]

2. Extract the square root of $8x+4+x^4+4x^3+8x^2$.
Root, x^2+2x+2.

3. Extract the square root of $25a^2x^2-12ax^3+16a^4+4x^4-24a^3x$.
Root, $2x^2-3ax+4a^2$.

4. Extract the square root of $x^6-6ax^5+15a^2x^4-20a^3x^3+15a^4x^2-6a^5x+a^6$.

5. Extract the square root of $x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}$.
Root, $x^2-\frac{x}{2}+\frac{2}{x}$.

6. Extract the square root of $9x-24x^{\frac{1}{2}}y^{\frac{2}{3}}+12x^{\frac{1}{2}}+16y^{\frac{4}{3}}-16y^{\frac{2}{3}}+4$.
Root, $3x^{\frac{1}{2}}-4y^{\frac{2}{3}}+2$.

7. Extract the square root of $9a^{-2}+12a^{-1}b^2-6a+4b^4-4a^2b^2+a^4$.
Root, $3a^{-1}+2b^2-a^2$.

Query.—In arranging the last with respect to a, why should $4b^4$ come before $-6a$?

208. **Scholium.**—Since the square root of a quantity is either + or —, all the signs in the above roots may be changed, and they will still be the roots of the same polynomials. Thus, in the 3rd Example, if we call the square root of $4x^4$, *minus* $2x^2$, $(-2x^2,)$ which it is, as well as $+2x^2$, and then continue the work as before, we get for the root $-2x^2+3ax-4a^2$.

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209. Prob. 3.—To extract the Square Root of a Decimal Number either exactly or approximately.

Rule.—*I. Separate the numbers into periods by placing a mark over units and over each alternate figure therefrom, calling the marked figure with the one at its left, if any, a period. The number of figures in the root is equal to the number of periods thus formed.*

II. Take the square root of the greatest square in the left hand period, and write it as the highest order in the root. Subtract the square of this figure from the period used, and to the remainder annex the next period for a new dividend.

III. Double the root already found for a Trial Divisor, by which divide the new dividend, rejecting in the trial the right hand figure of this dividend. The quotient is the next figure in the root or a greater one. To obtain the True Divisor annex to the Trial Divisor the last root figure. Multiply this True Divisor by the last root figure, subtract the product from the last new dividend, and to the remainder annex the next period of the given number for another new dividend.

IV. Double the root already found for a new Trial Divisor, and repeat the processes given in the 3rd paragraph till all the periods are brought down. If the number is a perfect square, the last remainder is *zero. If not, annex periods of two 0's each, and continue the process till the required degree of accuracy is attained. All the root figures arising from decimal fractional periods are decimal fractions.*

Scholium 1.—In separating decimal fractions, or the fractional part of mixed numbers into periods, make full periods of two figures each, annexing a 0 if necessary.

Scholium 2.—If at any time the Trial Divisor is not contained in the dividend to be used, annex a 0 to the root and also to the Trial Divisor, and then bring down the next period and divide.

Scholium 3.—When the work does not terminate with the last period of significant figures it will not terminate at all, and the given number is a surd. This is evident, since the process makes the unit's figure in each subtrahend, the square of the last figure in the root, but no figure squared gives 0 in unit's place. The process can, however, be carried to any given degree of accuracy.

Demonstration.—1st. That this method of pointing gives the number of places in the root, is made evident by squaring a few numbers. Thus the square of 1 is 1, and the square of 10 is 100, hence the squares of all numbers between 1 and 10 have 1 or 2 figures; that is, *Twice as many figures as the root, or one less than twice as many.* Again, the square of 100 is 10000; hence the square of all numbers between 10 and 100 have 3 or 4 figures; that is, *Twice as many as there are in the root, or one less.* In like manner it is readily seen that the *square of any* number consists of *Twice as many figures as the root, or one less.* Hence the method of pointing indicates the number of figures of which the root consists.

In the case of decimal fractions, since the number of decimals in a product equals the number in both the factors, there are always twice as many decimals in the square as in the root. Hence if the number of decimal places in the given number is odd, they are to be made even by annexing 0.

2nd. That the greatest square in the left hand period is the square of the highest order in the root, appears from the facts that the square of any number of *units* between 1 and 9 falls in the first right hand period, the square of any number of *tens* between 1 and 9 falls in the second period, the square of any number of *hundreds* between 1 and 9 falls in the third period, etc. Moreover, though the left hand period usually contains more than the square of the highest order in the root, *it can not* contain the square of a unit more of that order, since all the figures that can follow this highest order in the root can not make another unit of this order. Thus the square of 3999, can not be as great as the square of 4000; but the square of 4000 gives 16 in the highest period of the power, hence the square of 3999, must give less than 16 in that period.

3rd. To prove the method of finding and using the *Trial Divisor*, suppose, in any given case, the pointing shows that the root consists of 4 figures. Represent the thousands of the root by T, the hundreds by h, the tens by t, and the units by u. The number itself will be

$(T + h + t + u)^2 = T^2 + (2T+h)h+[2(T+h)+t]t+[2(T+h+t)+u]u$. (**207**, 3rd paragraph in the DEM.)

Whence having found and removed the square of the *thousands*, T^2, the next part of the power is $(2T+h)h$. But as the lowest order of this is hundreds multiplied by hundreds, or 10,000's we need not bring down anything below 10,000's, or the next period, for none of this part is contained in the lower periods. Again, for *trial*, considering this part as $2T \times h$, the product contains nothing lower than *hundred thousands;* hence the *ten thousands* figure is to be omitted in the trial division. But the true divisor is $2T+h$; hence the root figure is annexed to the trial divisor, or really *added* to it, regarding the local values.

4th. It is evident that this process is merely repeated as we increase the number of figures in the root; and as the law of notation is the same when we pass the decimal point into a fraction, no special exemplification is needed in such a case.

EXAMPLES.

Ex. 1. Extract the square root of 7284601.

Model Solution.

Operation.

$$\begin{array}{r l}
 & \dot{7}2\dot{8}4\dot{6}0\dot{1}\ (\ 2699 \\
 & \underline{4} \\
46\ ) & 328 \\
 & \underline{276} \\
529\ ) & 5246 \\
 & \underline{4761} \\
5389\ ) & 48501 \\
 & \underline{48501}
\end{array}$$

Explanation.—As the highest order in this number is millions, the highest order in the root is thousands, for the square of thousands is millions or ten millions, and the square of anything above thousands is more than ten millions. The root, therefore, consists of thousands, hundreds, tens, and units, as indicated by the pointing.

The square of the thousands figure in the root evidently must be sought in the 7 millions, or the left hand period. The greatest square in this being 4, the square root of which is 2, the thousands figure of the root is 2, which I therefore place in the root.

Now the root may be represented by $T+h+t+u$, T standing for the thousands, h for the hundreds, t for the tens, and u for the units. Hence the number itself will be represented by $(T+h+t+u)^2 = T^2+[2T+h]\,h+[2\,(T+h)+t]\,t+[2\,(T+h+t)+u]\,u$.

Having removed the T^2 (2 thousands squared) from the number, the second part is $(2T+h)h$, the lowest order in which is hundreds multiplied by hundreds, which gives ten thousands; wherefore I bring down no order lower than ten thousands, or simply the next period. As this new dividend contains $(2T+h)\,h$, I will, *for a trial*, consider it as simply containing $(2T)\times h$, and as hundreds into thousands produce only orders *above* ten thousands, I may omit the 8 which is ten thousands, in making this trial. Using $2T$ or 4 (thousands) for the trial divisor, I find it contained in 32 (hundred thousands) 8 times. But, as the trial divisor is too small by this new figure, it is evident that adding it, thus making the divisor 48, it will not be contained 8 times. Neither will it be contained 7 times. Thus I find 6 to be the next figure in the root, and the true divisor $2T+h$, to be 46 (*i. e.*, 4 thousands and 6 hundreds). Multiplying 46 by 6, and thus forming the part $[2T+h]\,h$, I find it to be 276 (ten thousands), which subtracted leaves 52 (ten thousands).

Again, these two parts, viz., T^2 and $(2T+h)\,h$ having been removed, the next part of the power is $[2\,(T+h)+t]\,t$, or [52 hundreds $+t]t$. As the lowest order of this part is tens squared, I need bring down nothing below hundreds, or the next period. The pupil can now fill out the demonstration as in the preceding paragraph.

2. Extract the square root of 7225. *Root,* 85.

3 to 7. Show that $\sqrt{9801} = 99$, $\sqrt{47089} = 217$, $\sqrt{553536} = 744$, $\sqrt{43046721} = 6561$, $\sqrt{5764801} = 2401$.

8 to 11. Show that $\sqrt{.5} = .7071+$, $\sqrt{3} = 1.73+$, $\sqrt{50} = 7.071+$, $\sqrt{5} = 2.236+$.

210. Cor.—*In extracting the roots of common fractions, if the numerator and denominator are perfect squares, extract the root of each separately; if they are not, it is usually best to convert the fraction into a decimal and then extract its root.*

12 to 15. $\sqrt{\frac{36}{49}} = \frac{6}{7}$, $\sqrt{\frac{5}{6}} = .912+$, $\sqrt{\frac{1}{3}} = .577+$, $\sqrt{\frac{2}{3}} = .8164+$.

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211. Prob. 4.—To extract the Cube Root of a Polynomial.

Rule.—*I. Arrange the polynomial with reference to one of its letters, as for division.*

II. Extract the cube root of the first left hand term. This root is the first term of the required root. Subtract the cube of this term of the root from the polynomial.

III. Take three times the square of the root already found for a Trial Divisor. By this trial divisor divide the first term of the remainder of the polynomial, and write the quotient as the second term of the root.

IV. Complete the divisor by adding to the trial divisor 3 times this last term multiplied by the part of the root previously found, and also the square of the last term found. Multiply the true divisor thus formed by the last term in the root, and subtract the product from the last remainder, bringing down such terms as may be necessary.

V. Repeat the process of forming Trial Divisors, dividing, completing the divisor, multiplying and subtracting, till the polynomial is exhausted, or until there is no term of it remaining which can be exactly divided by the first term of the Trial Divisor.

Demonstration.—1st. The polynomial is arranged as in division, since this is the order which the terms assume in cubing any polynomial, similarly arranged.

2nd. In cubing any polynomial, the first term of the cube is found to be the cube of the first term of the root; hence, in extracting the cube root, the cube root of this term is the first term of the root.

3d. To prove the process of finding the divisors and subsequent terms of the root, we observe the following operations:

$$\text{A.}\quad (a+b)^3 = a^3+3a^2b+3ab^2+b^3 = \overset{(1)}{a^3}+[3a^2+\overset{(2)}{3ab}+b^2]\,b.$$

$$\begin{aligned}
\text{B.}\quad (a+b+c)^3 &= [(a+b)+c]^3 \\
&= (a+b)^3+[3\,(a+b)^2+3\,(a+b)\,c+c^2]\,c \\
&= \overset{(1)}{a^3}+[3a^2+\overset{(2)}{3ab}+b^2]\,b+[3\,(a+b)^2+\overset{(3)}{3}\,(a+b)\,c+c^2]\,c.
\end{aligned}$$

$$\begin{aligned}
\text{C.}\quad (a+b+c+d)^3 &= [(a+b+c)+d]^3 \\
&= (a+b+c)^3+[3\,(a+b+c)^2+3\,(a+b+c)\,d+d^2]\,d \\
&= \overset{(1)}{a^3}+[3a^2\overset{(2)}{+}3ab+b^2]\,b+[3\,(a+b)^2+\overset{(3)}{3(a}+b)\,c+c^2]\,c \\
&\quad +\ [3\,(a+b+c)^2+\overset{(4)}{3}\,(a+b+c)\,d+d^2]\,d.
\end{aligned}$$

Hence it appears; 1st, That the cube of a polynomial is made up of as many parts as there are terms in the root; 2nd, that the *first* part is the cube of the first term of the root; 3d, That the *second* part is *three times the square of the first term of the root* + 3 *times the first term into the second term* + *the square of the second term, multiplied by the second term of the root;* 4th, That any one of the parts of the power, as the nth, is *Three times the square of the* $n-1$ *preceding terms of the root,* + 3 *times the product of these terms into the next, or* nth *term,* + *the square of this last or* nth *term, all these terms being multiplied by the last, or* nth *term of the root.*

Finally, it is evident that, if the work does not terminate by this process when the letter of arrangement disappears from the remainder, it can never terminate, since the divisor always contains this letter.

Scholium 1.—If the first term of the arranged polynomial is not a perfect cube the root cannot be extracted.

Scholium 2.—If at any time no term of the remainder is exactly divisible by the first term of the trial divisor, the root can not be extracted.

EXAMPLES.

Ex. 1. Extract the cube root of $36a^4c + 27a^4bc^2x - 8a^3 + b^3x^3 - 6ab^2x^2 + 27a^6c^3 - 36a^3bcx - 54a^5c^2 + 9a^2b^2cx^2 + 12a^2bx$.

Model Solution.—Operation.

$$27a^6c^3-54a^5c^2+36a^4c-8a^3+27a^4bc^2x-36a^3bcx+12a^2bx+9a^2b^2cx^2-6ab^2x^2+b^3x^3 \;\underline{|\; 3a^2c-2a+bx}$$

$$27a^6c^3$$

Trial Divisor:

$$3\times(3a^2c)^2 = 27a^4c^2 \qquad \Big|\; -54a^5c^2+36a^4c-8a^3$$

Corrections:

$$3(3a^2c)(-2a) = \quad -18a^3c$$

$$(-2a)^2 = \quad 4a^2$$

True Divisor:

$$27a^4c^2-18a^3c+4a^2 \;\Big|\; -54a^5c^2+36a^4c-8a^3$$

Trial Divisor:

$$3\,(3a^2c-2a)^2 = 27a^4c^2-36a^3c+12a^2 \qquad \Big|\; 27a^4bc^2x-36a^3bcx+12a^2bx+9a^2b^2cx^2-6ab^2x^2+b^3x^3$$

Corrections:

$$3\,(3a^2c-2a)\,bx = \quad 9a^2bcx-6abx$$

$$(bx)^2 = \quad b^2x^2$$

True Divisor:

$$27a^4c^2-36a^3c+12a^2+9a^2bcx-6abx+b^2x^2 \;\Big|\; 27a^4bc^2x-36a^3bcx+12a^2bx+9a^2b^2cx^2-6ab^2x^2+b^3x^3$$

Explanation.—1st. I arrange this polynomial with reference to a, and thus see at once the first two terms. But the terms $36a^4c$ and $27a^4bc^2x$ are of the same degree with respect to a, and hence to determine which is to have the precedence, I notice that the first term in the root will be $3a^2c$, and as the second term of the polynomial divided by 3 times the square of this gives the second term of the root, I observe that the terms containing a and c are all to have precedence over those containing b and x. Hence I write $36a^4c-8a^3$ next. The remaining terms I arrange, giving a the precedence and noticing that as x will be in the last term of the root, its higher powers will stand last.

2nd. As $27a^6c^3$ is the cube of the first term of the root, that term is $3a^2c$, which I consequently place in the root, and subtract the term $27a^6c^3$ from the polynomial.

3rd. As the second part of a cube of a polynomial is 3 times the square of the first term of the root, plus other terms, into the second term of the root, I take 3 times the square of this first term of the root or $27a^4c^2$, for a trial divisor. Dividing, I find the second term of the root to be $-2a$. But the *True Divisor*, or leading factor in this second part of the power, is 3 times the square of the former part of the root, + three times that part into the last term found, + the square of this term. Hence I add 3 times $3a^2c$ multiplied by $-2a$, and $-2a$ squared, to complete the divisor. Having completed it, I multiply it by the last term of the root found, $-2a$, and thus form the second part of the power of the root, which I subtract from the given polynomial.

4th. The explanations of the next and succeeding steps, when there are more, are identical with the last, and can be supplied by the student.

2. Extract the cube root of $a^3-8b^3+12ab^2-6a^2b$.
Root, $a-2b$.

3. Extract the cube root of $5x^3-1-3x^5+x^6-3x$.
Root, x^2-x-1.

4. Extract the cube root of $66x^4+1-63x^3-9x+8x^6-36x^5+33x^2$.
Root, $2x^2-3x+1$.

5. Extract the cube root of $60c^2x^4 + 48cx^5 - 27c^6 + 108c^5x - 90c^4x^2 + 8x^6 - 80c^3x^3$.

Root, $2x^2 + 4cx - 3c^2$.

6. Extract the cube root of $204c^4x^2 - 144c^5x + 8x^6 - 36cx^5 - 171c^3x^3 + 64c^6 + 102c^2x^4$.

Root, $2x^2 - 3cx + 4c^2$.

7. Extract the cube root of $27x - 8x^{\frac{3}{2}} - 36 + 36x^{\frac{5}{2}} + 12x^{-1} - 54x^{\frac{7}{2}} + 9x^{-\frac{5}{2}} + 27x^{\frac{9}{2}} + x^{-6} - 6x^{-\frac{7}{2}}$.

Operation.

$$\begin{array}{l|l}
 & 27x^{\frac{9}{2}} - 54x^{\frac{7}{2}} + 36x^{\frac{5}{2}} - 8x^{\frac{3}{2}} + 27x - 36 + 12x^{-1} + 9x^{-\frac{5}{2}} - 6x^{-\frac{7}{2}} + x^{-6} \;\underline{|\, 3x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-2}} \\
 & \underline{27x^{\frac{9}{2}}} \\
27x^3 - 18x^2 + 4x & - 54x^{\frac{7}{2}} + 36x^{\frac{5}{2}} - 8x^{\frac{3}{2}} \\
 & \underline{- 54x^{\frac{7}{2}} + 36x^{\frac{5}{2}} - 8x^{\frac{3}{2}}} \\
27x^3 - 36x^2 + 12x + 9x^{-\frac{1}{2}} - 6x^{-\frac{3}{2}} + x^{-4} & + 27x - 36 + 12x^{-1} + 9x^{-\frac{5}{2}} - 6x^{-\frac{7}{2}} + x^{-6} \\
 & \underline{+ 27x - 36 + 12x^{-1} + 9x^{-\frac{5}{2}} - 6x^{-\frac{7}{2}} + x^{-6}}
\end{array}$$

***212.* Prob. 5.—To extract the cube root of a decimal number, either exactly or approximately.**

Rule.—*I. Separate the number into periods by placing a mark over units and over each third figure therefrom, calling the figure marked, together with the two at its left (if there are so many), a period. The number of places in the root is the same as the number of periods.*

II. Take the cube root of the greatest cube in the left hand period, and write it as the highest order in the root. Subtract the cube of this figure from the period used, and to the remainder annex the next period for a new dividend.

III. Take three times the square of the root already found, regarding it as tens, for a trial divisor, by which divide the new dividend. The quotient is the next figure in the root (or a greater one). To obtain the True Divisor add to the trial divisor 3 times the product of the last root figure by the preceding part of the root, regarded as tens, and also the square of the last figure in the root. Multiply this true divisor by the last root figure and subtract the product from the last new dividend, and bring down the next period.

IV. Repeat this process till all the periods have been brought down. If the number is a perfect cube the remainder is zero. If not, annex periods of 3 zeros each and continue the operation till the required degree of accuracy is attained, marking the figures thus obtained as a decimal.

Scholium 1.—In pointing off decimal fractions, or the fractional part of mixed numbers, make full periods of three figures each, annexing 0's if necessary.

Scholium 2.—If at any time the trial divisor is not contained in the dividend to be used according to the 3rd paragraph in the rule, annex a 0 to the root and also two zeros to the trial divisor, bring down the next period, and then divide.

Scholium 3.—When the work does not terminate with the last period of significant figures it will not terminate at all, and the number is a surd. This is evident since the right hand figure in any subtrahend arises from cubing the corresponding digit in the root, and the cube of no digit produces 0 in unit's place.

Demonstration.—1st. That this method of pointing gives the number of figures in the root is made evident by cubing a few numbers. Thus the cube of 1 is 1, and of 10 is 1000; hence the cubes of all numbers between 1 and 10 have 1, 2, or 3 (cannot have 4) figures. The cube of 100 is 1,000,000; hence the cube of numbers between 10 and 100 have 4, 5, or 6 figures, but can not have 7. Again, the cube of 1000 is 1,000,000,000; hence the cube of any number between 100 and 1000, *i. e.*, of any number represented by 3 figures, contains 9, or *one or two* less than 9 figures. In like manner it appears that the cube of any integral number contains either *three times as many figures as the root, or one or two less.* In the multiplication of decimal fractions the number of fractional places in the product is equal to the number in both or all the factors used, hence the fractional part of any cube must have three times as many figures as the root.

2nd. That the greatest cube in the left hand period is the cube of the highest order in the root, appears from the facts that the cube of any number of *units* between 1 and 9 falls in the 1st period; the cube of any number of *tens* between 1 and 9, falls in the second; of any number of hundreds, in the 3rd, etc. Moreover, though the left hand period usually contains more than the cube of the digit in the highest order in the root, *it can not* contain the cube of a unit more of that order, since all the figures that can follow this highest order in the root can not make another unit of that order. Thus the cube of 3999 can not be as great as the cube of 4000. But the cube of 4000 gives 64 in the highest period. Hence the cube of 3999 must give less than 64 in that period.

3rd. In any given case, suppose the pointing shows that the root consists of thousands, hundreds, tens, and units. Represent the thousands by T, the hundreds by h, the tens by t, and the units by u. Then the number is $(T+h+t+u)^3 = T^3+[3T^2+3Th+h^2]h +[3(T+h)^2+3(T+h)t+t^2]t+[3(T+h+t)^2+3(T+h+t)u+u^2]u$. But having removed the cube of the thousands, T^3, the next part of the power is $[3T^2+3Th+h^2]h$. No part of this can fall in either of the two lowest periods of the power, since its lowest order arises from h^3 which is 1,000,000, at least. Hence we need only bring down one period. For a *trial*, we consider this part as $3T^2 \times h$, and hence the *Trial Divisor* is $3T^2$, or 3 *times the square of the root already found.* Again, regarding this thousands figure as tens, makes the T, which squared and multiplied by the next figure of the root which is also hundreds, give millions, the same order as the new dividend. But the *True Divisor* is $3T^2 + 3Th + h^2$; hence we add to $3T^2$, $3Th + h^2$; *i. e.*, 3 *times the root previously found multiplied by the last figure, and the square of this* last figure. In making this correction we are to remember to call the thousands so many *tens*, which reduces it to hundreds, the order of the root which we are seeking; whence the correction becomes the square of hundreds, or of the same order as the trial divisor, and can be added to it.

4th. It is evident that this process is merely repeated, as we proceed to obtain other figures in the root; and, as the law of notation is the same as we pass the decimal point, no special exemplification is needed in that case.

EXAMPLES.

Ex. 1. Extract the cube root of 99252847.

Model Solution.—Operation.

			99́252́847́	$\overset{h}{4}\ \overset{t}{6}\ \overset{u}{3}$
			64	
Trial Divisor	$3\,(40)^2$ =	4800	35252	
Corrections	$3\,(40)\,6$ =	720		
	6^2 =	36		
True Divisor	. . . .	5556	33336	
Trial Divisor	$3\,(460)^2$ =	634800	1916847	
Corrections	$3\,(460)\,3$ =	4140		
	$(3)^2$ =	9		
True Divisor	. . . .	638949	1916847	

Explanation.—As the highest order in this number is *ten millions*, the highest order in the root is *hundreds*, since the cube of a hundreds figure falls in millions period, while the cube of thousands falls in billions. Moreover, the cube of the hundreds figure is the greatest cube contained in 99, *i. e.* 64, the root of which is 4, which is, therefore, the hundreds of the root. That the hundreds figure is not *greater* than 4, is evident, since the cube of 5 hundreds is *greater* than the given number.

Therefore the cube root of the given number is $h+t+u$, and the number itself is $(h+t+u)^3 = h^3+[3h^2+3ht+t^2]t+[3(h+t)^2+(3h+t)u+u^2]u$. But having removed the h^3 by subtracting the 64 (millions), the next part of the power is $[3h^2+3ht+t^2]t$. Now the lowest order of this is t^3, or the cube of tens, which cannot fall below thousands, so that I need only bring down thousands period, *i. e.* the next lower. For a *Trial* I now consider this part of the power (35252) as $3h^2 \times t$, or $3[40]^2 \times t$. I reduce the 4 hundreds to the same order as the root figure which I am seeking, so that the product of its square by this root figure shall be of the same order as the new dividend. Therefore, reducing the 4 hundreds to tens it becomes 40, whence $3(40)^2 = 4800$, which being hundreds, goes into the new dividend, which is thousands, *tens* times. This *trial* divisor is really contained in the dividend 7 (*tens*) times, but as the corrections to be made upon it for the true divisor are so great, the *true* divisor is contained but 6 times, as I find by trying 7 for the tens of the root. Having thus found 6 to be the tens of the root, I correct my trial divisor, which by the formula is $3h^2+3h \times t+t^2$, by adding $3h \times t$ or $3(40) \times 6$, and t^2 or 6^2, and find the true divisor to be 5556 (hundreds). This multiplied by the 6 (tens) gives the second part of the power, *i. e.* $(3h^2+3ht+t^2)t = 33336$ (thousands), which I therefore subtract from the given number.

[The next step is exactly like the last, and the pupil can supply the demonstration.]

2. What is the cube root of 74088? *Ans.*, 42.

3. What is the cube root of 12326391? *Ans.*, 231.

4. What is the cube root of 122097755681? *Ans.*, 4961.

+**5.** What is the cube root of 2936.493568? *Ans.*, 14.32.

6. What is the cube root of 12.5? *Ans.*, 2.321 nearly.

7. What is the cube root of .64? *Ans.*, .8617+.

8. What is the cube root of .08? *Ans.*, .4308+.

9. What is the cube root of .008? *Ans.*, .2.

10 to **12.** Show that $\sqrt[3]{2} = 1.2599+$; $\sqrt[3]{5} = 1.7099+$; $\sqrt[3]{9} = 2.08008+$.

13 to **15.** Show that $\sqrt[3]{\frac{2}{3}} = .87+$; $\sqrt[3]{\frac{4394}{11664}} = \frac{13}{18}$; $\sqrt[3]{34\frac{21}{64}} = 3\frac{1}{4}$.

[**Note.**—With regard to the various shorter methods for extracting roots, the various methods of approximation and the like, no mathematician thinks of using them, or even those here given, but resorts at once to the table of logarithms. It is better that the student should spend his time in becoming perfectly familiar with the *demonstration* of a single method, than to cumber the memory with a multiplicity of processes which he will not remember, and which if we were to remember he would never use.]

FOR REVIEW OR ADVANCED COURSE.

***213.* Prob. 6.—To extract roots whose indices are composed of factors 2 and 3.**

Solution.—To extract the 4th root, extract the square root of the square root. Since the 4th root is one of the 4 equal factors into which a number is conceived to be resolved, if we first resolve a number into 2 equal factors (that is, extract the square root) and then resolve one of these factors into 2 equal factors (that is, extract its square root) one of the last factors is one of the 4 equal factors which compose the original number, and hence the 4th root. In like manner the 6th root is the cube root of the square root, etc.

EXAMPLES.

Ex. 1. What is the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$? *Ans.*, $\pm(2a-3x)$.

2. What is the 6th root of $15x^2-20x^3+x^6-6x^5+1-6x+15x^4$? *Ans.*, $\pm(x-1)$.

3. What is the 6th root of 2985984?

4. What is the 8th root of 1679616?

***214.* Prob. 7.—To extract the *m*th (any) root of a decimal number.**

Solution.—Any root can be extracted by a process altogether similar to those given for the square and cube roots, or by a simple inspection of the corresponding power of a binomial. Thus to extract the fifth root, point off by placing a point over units and every *fifth* figure therefrom, for the 7th root over every 7th figure, etc. Extract the required root of the largest power of the mth degree in the left hand period for the first figure in the root. Subtract, and bring down the next period. To form the trial and true divisors, and hence to find the other figures of the root, consider the corresponding power of a binomial. Thus for the 5th root, we have $(a+b)^5 = a^5+5a^4b+10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + [5a^4 + 10a^3b + 10a^2b^2+5ab^3]b$. The trial divisor is $5a^4$, *i. e.* five times the 4th power of the root already found regarded as *tens*. The corrections are $10a^3b+10a^2b^2+5ab^3$, regarding a as the root already found and as tens, and b as the next figure, *i. e.* the one sought by the trial.

In the 7th root the trial divisor is $7a^6$, and the corrections are $21a^5b+35a^4b^2+35a^3b^3+21a^2b^4+7ab^5+b^6$.

But in these cases, and much more in the case of higher roots, the trial divisor differs so much from the true divisor that the process is little better than guess-work.

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215. Prob. 8.—To extract the mth root of a polynomial.

Rule.—*I. Having arranged the polynomial as for division, take the root of the first term, for the first term of the required root.*

II. Subtract the power from the given quantity, and divide the first term of the remainder by the first term of the root involved to the next inferior power, and multiplied by the index of the given power; the quotient will be the next term of the root.

III. Subtract the power of the terms already found from the given quantity, and using the same divisor proceed as before.

Demonstration.—This rule demonstrates itself, as the final operation consists in involving the root to the required degree.

Scholium.—This rule may also be used for decimal numbers.

EXAMPLES.

Ex. 1. Find the fourth root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.

2. Find the fifth root of $x^5+5x^4+10x^3+10x^2+5x+1$.

3. Find the sixth root of $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

4. Extract the 5th root of 36936242722357.

Operation—(by 214).

Formula:	$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$		$369\acute{3}6242\acute{7}2235\acute{7}$	517
	$= a^5 + [5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4]\,b.$		3125	
Trial Divisor:	$5a^4 = 5\,(50)^4 =$	31250000	56862427	
Corrections: 1st.	$10a^3b = 10\,(50)^3 \times 1 =$	1250000		
2nd.	$10a^2b^2 = 10\,(50)^2 \times 1^2 =$	25000		
3rd.	$5ab^3 = 5\,(50) \times 1^3 =$	250		
4th.	$b^4 = 1^4 =$	1		
True Divisor:		32525251	32525251	
Trial Divisor:	$*5a^4 = 5\,(510)^4 =$	338260050000	2433717622357	
Corrections: 1st.	$10a^3b = 10\,(510)^3 \times 7 =$	9285570000		
2nd.	$10a^2b^2 = 10\,(510)^2 \times 7^2 =$	127449000		
3rd.	$5ab^3 = 5\,(510) \times 7^3 =$	874650		
4th.	$b^4 = 7^4 =$	2401		
True Divisor:		347673946051	2433717622357	

* In each operation a represents the part of the root before found, and b the figure sought by that operation.

5. What is the 5th root of 5? *Ans.*, 1.37974 nearly.

6. What is the 7th root of 1231171548132409344?

Operation.

Formula:	$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$		$1231\acute{1}71548\acute{1}32409344$	384
	$= a^7 + [7a^6 + 21a^5b + 35a^4b^2 + 35a^3b^3 + 21a^2b^4 + 7ab^5 + b^6]\,b.$		2187	
Trial Divisor:	$7a^6 = 7\,(30)^6 =$	5103000000	1012471548132	
Corrections:	1st. $21a^5b = 21\,(30)^5 \times 8 =$	4082400000		
	2nd. $35a^4b^2 = 35\,(30)^4 \times 8^2 =$	1814400000		
	3rd. $35a^3b^3 = 35\,(30)^3 \times 8^3 =$	483840000		
	4th. $21a^2b^4 = 21\,(30)^2 \times 8^4 =$	77414400		
	5th. $7ab^5 = 7\,(30) \times 8^5 =$	6881280		
	6th. $b^6 = 8^6 =$	262144		
		11568197824	92545582592	
Trial Divisor:	$7a^6 = 7\,(380)^6 =$	210765546880000000	870157222124093444	
Corrections:	1st. $21a^5b = 21\,(380)^5 \times 4 =$	66557541120000		
	2nd. $35a^4b^2 = 35\,(380)^4 \times 4^2 =$	1167676160000		
	3rd. $35a^3b^3 = 35\,(380)^3 \times 4^3 =$	12291328000		
	4th. $21a^2b^4 = 21\,(380)^2 \times 4^4 =$	776294400		
	5th. $7ab^5 = 7\,(380) \times 4^5 =$	2723840		
	6th. $b^6 = 4^6 =$	4096		
		217539305531023336	870157222124093444	

4. What is the 7th root of 2?

Ans., 1.104+.

216. **General Scholium.**—The student should learn to see the rule for the square root in $(a+b)^2 = a^2 + (2a+b)b$; for the cube root in $(a+b)^3 = a^3 + (3a^2 + 3ab + b^2)\,b$; for the fifth root in $(a+b)^5 = a^5 + (5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4)\,b$, etc.

CALCULUS OF RADICALS

SECTION III.

REDUCTIONS.

217. **Prob. 1.—To simplify a radical by removing a factor.**

Rule.—*Resolve the number under the radical sign into two factors, one of which shall be a perfect power of the degree of the radical. Extract the required root of this factor and place it before the radical sign as a coefficient to the other factor under the sign.*

Demonstration.—This process is simply an application of Cor. Art. **205,** which proves that the product of the roots is equal to the root of the product. Thus $\sqrt{48a^5b^3} = \sqrt{16a^4b^2 \times 3ab} = \sqrt{16a^4b^2} \times \sqrt{3ab} = 4a^2b\sqrt{3ab}$.

EXAMPLES.

Ex. 1.—Reduce to its simplest form $\sqrt[3]{189a^4b^3c^2}$.

Model Solution.

Operation. $\sqrt[3]{189a^4b^3c^2} = \sqrt[3]{27a^3b^3 \times 7ac^2} = \sqrt[3]{27a^3b^3} \times \sqrt[3]{7ac^2} = 3ab\sqrt[3]{7ac^2}$.

Explanation. $\sqrt[3]{189a^4b^3c^2}$ indicates that $189a^4b^3c^2$ is to be resolved into 3 equal factors. Therefore I first separate it into two factors, $27a^3b^3$ and $7ac^2$, one of which is a perfect cube. Now I resolve each of these two factors into 3 equal factors, and taking one of each three I multiply them together. This product will constitute one of the three equal factors of the given number. $27a^3b^3 = 3ab \times 3ab$

$\times 3ab$, and $7ac^2 = \sqrt[3]{7ac^2} \times \sqrt[3]{7ac^2} \times \sqrt[3]{7ac^2}$. Hence $3ab\sqrt[3]{7ac^2}$ is one of the 3 equal factors of $189a^4b^3c^2$; or $\sqrt[3]{189a^4b^3c^2} = 3ab\sqrt[3]{7ac^2}$.

2. Reduce $\sqrt{27a^3x^5}$ to its simplest form.

Result, $3ax^2\sqrt{3ax}$.

3. Reduce $\sqrt{80a^3x^5}$ to its simplest form.

4. Reduce $\sqrt[3]{375x^4y^3}$ to its simplest form.

Suggestion.—If the factor of the decimal number under the radical is not apparent, it can readily be found by a few trials. Thus in the 3rd, we could *try* the square of 2, or 4; and then the square of 3, or 9; and then of 4, or 16; of 5, or 25; of 6, or 36; of 7, or 49; of which we should find 16 to be the greatest square factor. We need not try farther than 49, since this is more than $\frac{1}{2}$ of 80, and no larger number can be a factor.

5. Simplify $\sqrt{1183x^5y^3}$.

Suggestion.—Try the square numbers from 4 upward till you find the required factor. But a little judgment will save labor. Thus, we need not try 4, for no number multiplied by 4 gives a 3 in units' place. For a like reason we *would* try 9, but not 16, 25 or 36. Then again we would try 49, but not 64; 81, but not 100; 121, but not 144. Finally 169 meets the case, and we have $\sqrt{1183x^5y^3} = \sqrt{169x^4y^2 \times 7xy} = \sqrt{169x^4y^2} \times \sqrt{7xy} = 13x^2y\sqrt{7xy}$.

6. Simplify $\sqrt{3179a^5bx^7}$.

7. Simplify $\sqrt[5]{96a^7x^9}$. *Result*, $2ax\sqrt[5]{3a^2x^4}$.

8. Simplify $\sqrt[3]{7290x^{3m}y^{6n+2}}$. *Result*, $9x^my^{2n}\sqrt[3]{10y^2}$.

9. Simplify $(1715a^{9m-n}b^{6n})^{\frac{1}{3}}$.

Result, $7a^{3m}b^{2n}\sqrt[3]{5a^{-n}}$, or $7a^{3m}b^{2n}\left[\frac{5}{a^n}\right]^{\frac{1}{3}}$.

10. Simplify $(352a^{\frac{5}{6}}b^{10})^{\frac{1}{5}}$. *Result*, $2a^{\frac{1}{6}}b^2\sqrt[5]{11}$.

Scholium.—Of course, by the use of fractional exponents, all the factors of such monomials may be written separately, as in (**202**). Thus, the result in *Ex.* 7, may be written $(96)^{\frac{1}{5}}a^{\frac{7}{5}}x^{\frac{9}{5}}$, or $2(3)^{\frac{1}{5}}aa^{\frac{2}{5}}xx^{\frac{4}{5}}$, or $2ax(3)^{\frac{1}{5}}a^{\frac{2}{5}}x^{\frac{4}{5}}$, or, by taking the 5th root of the product instead of the product of the 5th roots of the last 3 factors, $2ax(3a^2x^4)^{\frac{1}{5}}$, as above. The method of this rule is usually applied for removing factors which can be expressed without fractional exponents.

11. Simplify $\sqrt{a^3-a^2x}$. *Result*, $a\sqrt{a-x}$.

12. Simplify $\sqrt[n]{a^{m+n}b^n}$. *Result*, $ab\sqrt[n]{a^m}$.

13. Simplify $\sqrt{(a^2-b^2)(a+b)}$. *Result*, $(a+b)\sqrt{a-b}$.

14. Simplify $3\sqrt{50x^5}$. *Result*, $15x^2\sqrt{2x}$.

Suggestion.—When the radical has a coefficient, the factor removed from under the radical sign is to be multiplied into this coefficient.

15. Simplify $(x+y)\sqrt{x^3-2x^2y+xy^2}$.
Result, $(x^2-y^2)\sqrt{x}$.

16. Simplify $xy\sqrt{x^3y^2-x^2y^3}$. *Result*, $x^2y^2\sqrt{x-y}$.

17. Simplify $\sqrt[5]{x^{10}y^{-5}z^{5m+1}}$. *Result*, $x^2y^{-1}z^m\sqrt[5]{z}$.

18. Reduce $\sqrt{\frac{2}{3}}$ to its simplest form. *Result*, $\frac{1}{3}\sqrt{6}$.

Scholium.—A surd fraction is conceived to be in its simplest form when the smallest possible *whole number* is left under the radical sign. The reason for this is, not only that the radical factor is thus made simpler, but, if a fraction were to be left under the radical sign the question would arise, What fraction? Certainly not the least possible, for such a fraction can be diminished at pleasure. Thus, $\sqrt{\frac{2}{3}}=\sqrt{4\times\frac{1}{6}}=2\sqrt{\frac{1}{6}}=2\sqrt{16\times\frac{1}{96}}=8\sqrt{\frac{1}{96}}$, etc., without limit. Perhaps, if a fraction is to be left under a radical sign it will be proper to consider the expression as simplest when the fraction is nearest unity; whence $\sqrt{\frac{2}{3}}$ is to be considered as simpler than $8\sqrt{\frac{1}{96}}$.

218. Cor.—*The denominator of a surd fraction can always be removed from under a radical sign by multiplying both terms of the fraction by some factor which will make the denominator a perfect power of the degree required.*

Thus, $\sqrt[3]{\frac{5}{6}}=\sqrt[3]{\frac{36\times5}{36\times6}}=\sqrt[3]{\frac{180}{216}}=\sqrt[3]{\frac{1}{216}\times180}=\frac{1}{6}\sqrt[3]{180}$.

19. Reduce $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{3}}$, and $\sqrt[3]{\frac{1}{2}}$ to their simplest forms.

Scholium.—The root of a fraction having 1 for its numerator is equal to the same fraction into the root of the next lower power of the denominator. Thus, $\sqrt{\frac{1}{7}}=\frac{1}{7}\sqrt{7}$, $\sqrt[3]{\frac{1}{7}}=\frac{1}{7}\sqrt[3]{49}$, $\sqrt[4]{\frac{1}{7}}=\frac{1}{7}\sqrt[4]{343}$, etc.

20. Reduce $\sqrt{\frac{5}{12}}$ to its simplest form.

21. Reduce $\left[\frac{a-x}{a+x}\right]^{\frac{1}{2}}$ to its simplest form.

Result, $\frac{1}{a+x}\sqrt{a^2-x^2}$.

22. Reduce $\frac{3}{11}\sqrt{\frac{4}{7}}$, $\frac{1}{2}\sqrt{\frac{x^3}{15}}$, and $4\sqrt{\frac{3a^3}{8b}}$ to their simplest forms.

Results, (not in order), $\frac{x}{30}\sqrt{15x}$, $\frac{a}{b}\sqrt{6ab}$, and $\frac{6}{77}\sqrt{7}$.

23. Reduce $\frac{a}{a^2-b^2}\sqrt{\frac{a^3b-2a^2b^2+ab^3}{b^2}}$ to its simplest form.

Result, $\frac{a}{b\,(a+b)}\sqrt{ab}$.

24. Reduce $\sqrt[6]{\frac{1}{2}}$, and $\sqrt[5]{\frac{3}{4}}$ to their simplest forms.

Scholium.—The above simplifications can always be effected upon fractions, but upon integral radicals only when the integer has a factor which is a perfect power of the degree of the radical.

219. Prob. 2.—To simplify a radical, or reduce it to its lowest terms, when the index is a composite number, and the number under the radical sign is a perfect power of the degree indicated by one of the factors of the index.

Rule.—*Extract that root of the number which corresponds to one of the factors of the index, and write this root as a surd of the degree of the other factor of the given index.*

Demonstration.—The *mn*th root is one of the *mn* equal factors of a number. If, now, the number is resolved first into *m* equal factors, and then one of these *m* factors is again resolved into *n* other equal factors, one of the latter is the *mn*th root of the number.

Illustration.—The 4th root of a number is one of the four equal factors of that number; if we resolve the number into 2 equal factors, and then one of these factors into 2 other equal factors, one of the latter is one of the 4 equal factors which compose the given number.

EXAMPLES.

Ex. 1. Reduce $\sqrt[4]{25a^4b^6}$.

Model Solution.

Operation. $\sqrt[4]{25a^4b^6} = \sqrt{\sqrt{25a^4b^6}} = \sqrt{5^2b^3} = ab\sqrt{5b}.$

Explanation.—The 4th root of $25a^4b^6$ is one of the 4 equal factors of it. Hence I first resolve it into two equal factors, one of which is $5a^2b^3$. Then I resolve $5a^2b^3$ into two equal factors, or rather indicate it, as the operation cannot be fully performed, and have $\sqrt{5a^2b^3}$. $\sqrt{5a^2b^3}$ is, therefore, one of the 4 equal factors of $25a^4b^6$. But, by the last problem, $\sqrt{5a^2b^3} = ab\sqrt{5b}$. Hence $\sqrt[4]{25a^4b^3} = ab\sqrt{5b}$.

2. Reduce $\sqrt[6]{27a^3b^9}$. *Result,* $b\sqrt{3ab}$.

Suggestions. $\sqrt[6]{27a^3b^9} = \sqrt[2]{\sqrt[3]{27a^3b^9}} = \sqrt{3ab^3} = b\sqrt{3ab}.$

3. Reduce $\sqrt[6]{-64a^3}$. *Result,* $2\sqrt{-a}$.

4. Reduce $\sqrt[8]{256a^{12}x^8}$.

5. Reduce $\sqrt[4]{81n^2m^4}$.

6. Reduce $\sqrt[4]{x^2-2xy+y^2}$.

***220.* Prob. 3.**—**To reduce any number to the form of a radical of a given degree.**

Rule.—*Raise the number to a power of the same degree as the radical, and place this power under the radical sign with the required index, or indicate the same thing by a fractional exponent.*

Demonstration.—That this process does not change the value of the expression is evident, since the number is first involved to a given power, and then the corresponding root of this power is indicated, the latter, or *indicated* operation, being just the reverse of the former. Thus, $x = \sqrt[m]{x^m}$. That is, raising x to the mth power, and then indicating the mth root, leaves the value represented unchanged.

EXAMPLES.

Ex. 1. Reduce $7a^2x^3$ to a form of a radical of the 3rd degree.

Operation. $7a^2x^3 = \sqrt[3]{(7a^2x^3)^3} = \sqrt[3]{343a^6x^9}$.

Explanation.—If I cube $7a^2x^3$ and then extract the cube root of this cube, the result will evidently be the same as at first. Now $(7a^2x^3)^3 = 343a^6x^9$. But instead of *performing* the operation of extracting the cube root of $343a^6x^9$, which would evidently return it to $7a^2x^3$, I simply *indicate* the operation, and have $\sqrt[3]{343a^6x^9}$.

2. Reduce $2ay-3$ to the form of a radical of the second degree. *Result,* $\sqrt{4a^2y^2-12ay+9}$.

3. Reduce $a-x$ to the form of the cube root.

4. Reduce $\sqrt{\frac{2}{3}}$ to the form of the 4th root.

Sug. $\sqrt{\frac{2}{3}} = \sqrt[4]{\left(\sqrt{\frac{2}{3}}\right)^4} = \sqrt[4]{\sqrt{\frac{2}{3}} \times \sqrt{\frac{2}{3}} \times \sqrt{\frac{2}{3}} \times \sqrt{\frac{2}{3}}}$
$= \sqrt[4]{\frac{2}{3} \times \frac{2}{3}} = \sqrt[4]{\frac{4}{9}}$.

5. Reduce $\frac{3}{5}$ to the form of the 3rd root.

Result, $\sqrt[3]{\frac{27}{125}}$.

6. Reduce $\frac{x^2}{3}$ to the form of the 4th root.

221. Cor.—*To introduce the coefficient of a radical under the radical sign, it is necessary to raise it to a power of the same degree as the radical; for the coefficient being reduced to the same form as the radical by the last rule, we have the product of two like roots, which is equal to the root of the product* (**194**, *and* **205**).

EXAMPLES.

1. Introduce the coefficient in $3x\sqrt[3]{2x^2}$ under the radical sign.

Model Solution.

Operation. $3x\sqrt[3]{2x^2} = \sqrt[3]{27x^3} \times \sqrt[3]{2x^2} = \sqrt[3]{27x^3 \times 2x^2} = \sqrt[3]{54x^5}$.

Explanation.—Cubing $3x$ I have $27x^3$, the indicated cube root of which is $\sqrt[3]{27x^3}$. This is evidently the same in value as $3x$. Hence $3x\sqrt[3]{2x^2} = \sqrt[3]{27x^3} \times \sqrt[3]{2x^2} = \sqrt[3]{27x^3 \times 2x^2} = \sqrt[3]{54x^5}$ (**209**).

2. Introduce the coefficient in $\frac{1}{2}\sqrt{2}$ under the radical sign; in $\frac{1}{3}\sqrt{3}$; in $\frac{1}{2}\sqrt[3]{4}$; in $\frac{1}{3}\sqrt[3]{9}$.

Results, $\sqrt[3]{\frac{1}{2}}$, $\sqrt{\frac{1}{3}}$, $\sqrt[3]{\frac{1}{3}}$, $\sqrt{\frac{1}{2}}$.

3. Introduce the coefficients in the following expressions under the radical signs; $3a^2\sqrt{2ax}$, $(a-x)\sqrt{a+x}$, $\frac{1}{2}\sqrt{4a}$, $(x-y)\sqrt{x-y}$, and $\frac{c^2}{3}\sqrt[3]{27a^2}$.

Two of the results are $\sqrt{(a^2-x^2)(a-x)}$, and $\sqrt{(x-y)^3}$.

4. Introduce under the radical signs the coefficient in the following: $\frac{1}{2}\sqrt[4]{x^2}$, $\frac{3}{2x}\sqrt[3]{\frac{9}{4x^2}}$, $5x\sqrt[m]{25x^{-2}}$, and $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}$.

The last two are $\sqrt[m]{5^{m+2}x^{m-2}}$, and $\sqrt{\frac{a+b}{a-b}}$.

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222. Prob. 4.—To reduce radicals of different degrees to equivalent ones having a common index.

Rule.—*Represent the numbers by means of fractional indices. Reduce the indices to forms having a common denominator. Perform upon the numbers the operations represented by the numerators, and indicate the operation signified by the denominator.*

Demonstration.—The only point in this rule needing further demonstration is, that multiplying numerator and denominator of a fractional index by the same number does not change the value of the expression, *i. e.*, that $x^{\frac{a}{b}} = x^{\frac{ma}{mb}}$. Now, $x^{\frac{a}{b}} = x^{\frac{1}{b}} \times x^{\frac{1}{b}} \times x^{\frac{1}{b}}$ - - - - to a equal factors. If now we resolve each of these factors into m equal factors, x will be resolved into mb equal factors, and one of them will be represented $x^{\frac{1}{mb}}$. But as in $x^{\frac{a}{b}}$ there are a factors, each $x^{\frac{1}{b}}$, and as each of these is resolved into m factors, there will be in all ma factors, each $x^{\frac{1}{mb}}$. Hence $x^{\frac{a}{b}} = x^{\frac{ma}{mb}}$.

[When $\frac{3}{4}$ is used as a common fraction, we show that $\frac{3}{4} = \frac{6}{8}$ thus: $\frac{3}{4}$ signifies 3 of the 4 equal *parts* of some quantity. If now we separate each of these 3 *parts* into 2, the entire quantity will be separated into 8 *parts*, and in the three parts there will be 6. $\therefore \frac{3}{4} = \frac{6}{8}$. In an *analogous* manner when $\frac{3}{4}$ is used as an exponent, we show that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$, thus: $\frac{3}{4}$ as an exponent signifies 3 of the 4 equal *factors* of some quantity. Now if we resolve each of these 3 equal *factors* into 2, the entire quantity will be resolved into 8 *factors*, and in the 3 *factors* there will be 6. $\therefore \frac{3}{4}$ as an exponent is equivalent to $\frac{6}{8}$.]

EXAMPLES.

Ex. 1. Reduce $\sqrt{2a^3x}$ and $\sqrt[3]{4m^2y}$ to forms having a common index.

Model Solution.

Operation. $\sqrt{2a^3x} = (2a^3x)^{\frac{1}{2}}$, and $\sqrt[3]{4m^2y} = (4m^2y)^{\frac{1}{3}}$. But $(2a^3x)^{\frac{1}{2}} = (2a^3x)^{\frac{3}{6}} = (8a^9x^3)^{\frac{1}{6}} = \sqrt[6]{8a^9x^3}$; and $(4m^2y)^{\frac{1}{3}} = (4m^2y)^{\frac{2}{6}} = \sqrt[6]{16m^4y^2}$.

Explanation.—Representing the given numbers by means of fractional indices, I have $(2a^3x)^{\frac{1}{2}}$ and $(4m^2y)^{\frac{1}{3}}$. These indices reduced to forms having a common denominator are $\frac{3}{6}$ and $\frac{2}{6}$. Now $(2a^3x)^{\frac{1}{2}}$ signifies *one* of the *two* equal factors of $2a^3x$; while $(2a^3x)^{\frac{3}{6}}$ signifies *three* of the *six* equal factors of $2a^3x$. Hence $(2a^3x)^{\frac{1}{2}} = (2a^3x)^{\frac{3}{6}}$. In like manner $(4m^2y)^{\frac{1}{3}}$ signifies *one* of the *three* equal factors of $4m^2y$; while $(4m^2y)^{\frac{2}{6}}$ signifies *two* of the *six* equal factors of the same. Hence $(4m^2y)^{\frac{1}{3}} = (4m^2y)^{\frac{2}{6}}$. Finally, as $(2a^3x)^{\frac{3}{6}}$ is the same as the 6th root of the cube of $2a^3x$, I have $\sqrt[6]{8a^9x^3}$. In like manner $(4m^2y)^{\frac{2}{6}}$, meaning the 6th root of the square of $4m^2y$, becomes $\sqrt[6]{16m^4y^2}$.

2. Reduce $\sqrt{2}$ and $\sqrt[3]{3}$ to forms having a common index.

Results, $\sqrt[6]{8}$ and $\sqrt[6]{9}$.

3. Reduce $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{2}$ to forms having a common index.

4. Reduce $\sqrt[3]{2a^2}$, $\sqrt[5]{3x}$, and $\sqrt{xy}$ to forms having a common index. *One result is* $\sqrt[30]{729x^6}$.

5. Reduce $\sqrt{ax}$ and $\sqrt[3]{bx^2}$ to forms having a common index. *Results*, $(a^3x^3)^{\frac{1}{6}}$ and $(b^2x^4)^{\frac{1}{6}}$.

6. Reduce $a^{\frac{1}{4}}$, $(5b)^{\frac{1}{6}}$ and $(3c)^{\frac{1}{8}}$ to forms having a common index. *Results*, $\sqrt[24]{27c^3}$, $\sqrt[24]{a^6}$, $\sqrt[24]{625b^4}$.

7. Reduce $x^{\frac{1}{m}}$ and $y^{\frac{1}{n}}$ to forms having a common index.

8. Reduce $2c\sqrt{x}$ and $5a\sqrt[3]{2y}$ to forms having a common index.

Suggestion.—The radical factors can be reduced to forms having a common index without affecting the coefficient, since the operation does not affect the value of the radical.

9. Reduce $4\sqrt[3]{5x^2y}$, $2\sqrt[6]{2xy}$, and $10a\sqrt{3bx}$ to forms having a common index. *Results,* $2\sqrt[6]{2xy}$, $10a\sqrt[6]{27b^3x^3}$, $4\sqrt[6]{25x^4y^2}$.

10. Reduce $a+c$ and $(a-c)^{\frac{1}{2}}$ to forms having a common index. *Results,* $(a^2+2ac+c^2)^{\frac{1}{2}}$ and $(a-c)^{\frac{1}{2}}$.

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223. Prob. 5.—To reduce a fraction having a monomial radical denominator, or a monomial radical factor in its denominator, to a form having a rational denominator.

Rule.—*Represent the radical with a fractional index, and then multiply both terms of the fraction by the quantity in the denominator with an index which added to the given index makes it integral.*

Demonstration.—Since two factors consisting of the same quantity affected by the same or different exponents are multiplied by adding the exponents (**90**), and the sum of the exponent of the denominator and the factor by which we multiply it is an integer, the product becomes rational. The value of the fraction is not altered, since both its terms are multiplied by the same number.

EXAMPLES.

Ex. **1.** Reduce $\frac{3a\sqrt{2x}}{\sqrt{3x}}$ to a form having a rational denominator.

Model Solution.

Operation. $\frac{3a\sqrt{2x}}{\sqrt{3x}}=\frac{3a(2x)^{\frac{1}{2}}\times(3x)^{\frac{1}{2}}}{(3x)^{\frac{1}{2}}\times(3x)^{\frac{1}{2}}}=\frac{3a(6x^2)^{\frac{1}{2}}}{3x}=a\sqrt{6}.$

Explanation.—Using fractional exponents, I have $\frac{3a(2x)^{\frac{1}{2}}}{(3x)^{\frac{1}{2}}}$. Since multiplying numerator and denominator by the same number does not alter the value of the fraction, and as $(3x)^{\frac{1}{2}}\times(3x)^{\frac{1}{2}}$ makes $3x$, I can rationalize the denominator of this fraction by multiplying both its terms by $(3x)^{\frac{1}{2}}$. This gives $\frac{3a(6x^2)^{\frac{1}{2}}}{3x}$ which reduces to $a\sqrt{6}$. [If the rationale of these last reductions is not perfectly familiar it should be given. Thus $(6x^2)^{\frac{1}{2}}=(6)^{\frac{1}{2}}(x^2)^{\frac{1}{2}}=6^{\frac{1}{2}}x$, whence $\frac{3a(6x^2)^{\frac{1}{2}}}{3x}=\frac{3a(6)^{\frac{1}{2}}x}{3x}$, and cancelling the $3x$ I have $a\sqrt{6}$.]

2. Rationalize the denominator of $\frac{2}{3\sqrt{a^3}}$.

Suggestion. $\frac{2}{3\sqrt{a^3}}=\frac{2}{3a^{\frac{3}{2}}}$, and $a^{\frac{1}{2}}$ is the factor which rationalizes it.

Result, $\frac{2\sqrt{a}}{3a^2}$.

3. Rationalize the denominator of $\frac{b}{\sqrt{a}}$.

4. Rationalize the denominator of $\frac{b}{\sqrt[3]{a}}$. *Result,* $\frac{b\sqrt[3]{a^2}}{a}$.

5. Reduce $\frac{b}{\sqrt[n]{a}}$ to a form having a rational denominator.

Result, $\frac{b\sqrt[n]{a^{n-1}}}{a}$.

6. Reduce $\frac{1}{\sqrt{3}}$, $\frac{2}{\sqrt[3]{3}}$, $\frac{6}{\sqrt[5]{64}}$, and $\frac{\sqrt{3}}{\sqrt{6}}$ to forms having rational denominators. *One of the results is* $\frac{3}{2}\sqrt[5]{16}$.

Suggestion. $\frac{\sqrt{3}}{\sqrt{6}}=\frac{(3)^{\frac{1}{2}}\times(6)^{\frac{1}{2}}}{(6)^{\frac{1}{2}}\times(6)^{\frac{1}{2}}}=\frac{\sqrt{18}}{6}=\frac{3\sqrt{2}}{6}=\frac{1}{2}\sqrt{2}$.

Scholium.—This process is equally applicable to any form of *radical factor* in the denominator, whether monomial or polynomial.

7. Rationalize the denominator of $\frac{\sqrt{a-x}}{\sqrt{a+x}}$.

Suggestion. $\frac{\sqrt{a-x}}{\sqrt{a+x}}=\frac{(a-x)^{\frac{1}{2}}\times(a+x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}\times(a+x)^{\frac{1}{2}}}=\frac{\sqrt{a^2-x^2}}{a+x}$.

8. Reduce $\frac{3a}{2\sqrt[3]{3c-x^2}}$ to a form having a rational denominator.

Result, $\frac{3a\sqrt[3]{(3c-x^2)^2}}{6c-2x^2}$.

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224. Prob. 6.—To rationalize the denominator of a fraction when it consists of a binomial, one or both of whose terms are radicals of the second degree.

Rule.—*Multiply both terms of the fraction by the denominator with one of its signs changed.*

Demonstration.—This rationalizes the denominator, since in any case it gives the product of the sum and difference of the two terms of the denominator, which being equal to the difference of their squares, frees either or both from radicals, as the square of a square root is rational.

EXAMPLES.

Ex. 1. Rationalize the denominator of $\frac{a}{a-\sqrt{b}}$.

Model Solution.

Operation. $\frac{a}{a-\sqrt{b}}=\frac{a(a+\sqrt{b})}{(a-\sqrt{b})(a+\sqrt{b})}=\frac{a^2+a\sqrt{b}}{a^2-b}$.

Explanation.—I observe that $a-\sqrt{b}$ will be rationalized by multiplying it by $a+\sqrt{b}$, since the product of the sum and difference of two quantities is the difference of their squares. Hence multiplying *both terms* of the fraction, so as not to alter its value, I have $\frac{a^2+a\sqrt{b}}{a-b}$

2. Rationalize the denominator of $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

Result, $\frac{a+2\sqrt{ab}+b}{a-b}$.

3. Reduce $\frac{3}{2\sqrt{2}-3\sqrt{3}}$ to a form having a rational denominator.

Result, $\frac{3(2\sqrt{2}+3\sqrt{3})}{8-27}=\frac{6\sqrt{2}+9\sqrt{3}}{-19}$ or $-\frac{6\sqrt{2}+9\sqrt{3}}{19}$.

4. Reduce $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$ to a form having a rational denominator. *Result*, $4+\sqrt{2}$.

5. Rationalize the denominators of $\frac{1-\sqrt{5}}{3+\sqrt{5}}$, $\frac{1+\sqrt{2}}{2+\sqrt{2}}$, $\frac{3\sqrt{5}-2\sqrt{2}}{2\sqrt{5}-\sqrt{18}}$, and $\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

Results, $2-\sqrt{5}$, $\frac{1}{2}\sqrt{2}$, $9+\frac{5}{2}\sqrt{10}$, and $\frac{a-\sqrt{a^2-x^2}}{x}$.

6. Rationalize the denominator of

$$\frac{\sqrt{x^2+x+1}-\sqrt{x^2-x-1}}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}.$$

Result, $\frac{x^2-\sqrt{x^4-x^2-2x-1}}{x+1}$.

FOR REVIEW OR ADVANCED COURSE.

***225.* Prob. 7.—A factor may be found which will rationalize any binomial radical.**

Demonstration.—If the binomial radical is of the form $\sqrt[n]{(a+b)^m}$, or $(a+b)^{\frac{m}{n}}$, the factor is $(a+b)^{\frac{n-m}{n}}$, according to (**223**).

If the binomial is of the form $\sqrt[m]{a^s}+\sqrt[n]{b^r}$ or $a^{\frac{s}{m}}+b^{\frac{r}{n}}$, let $a^{\frac{1}{m}}=x$, and $b^{\frac{1}{n}} = y$; whence $a^{\frac{s}{m}} = x^s$, and $b^{\frac{r}{n}} = y^r$. Also let p be the least common multiple of m and n, whence x^{sp} and y^{rp} are rational. But $x^{sp} = a^{\frac{sp}{m}}$, and $y^{rp} = b^{\frac{rp}{n}}$. If now we can find a factor which will render $x^s + y^r$, $x^{sp} \pm y^{rp}$, this will be a factor which will render $a^{\frac{s}{m}} + b^{\frac{r}{n}}$, $a^{\frac{sp}{m}} \pm b^{\frac{rp}{n}}$ which is rational. To find the factor which multiplied by $x^s + y^r$ gives $x^{sp} \pm y^{rp}$, we have only to divide the latter by the former. Now $\frac{x^{sp} \pm y^{rp}}{x^s + y^r} = x^{s(p-1)} - x^{s(p-2)}y^r + x^{s(p-3)}y^{2r} - x^{s(p-4)}y^{3r} + \text{- - -} \pm y^{r(p-1)}$ (A), the + sign of the last term to be taken when p is odd, and the − sign when it is even (**126**). Therefore $x^{s(p-1)} - x^{s(p-2)}y^r + x^{s(p-3)}y^{2r} - x^{s(p-4)}y^{3r} + \text{- - -} \pm y^{r(p-1)}$, is a factor which will render $\sqrt[m]{a^s} + \sqrt[n]{b^r}$ rational, x^s being understood to be $a^{\frac{s}{m}}$, and $y^r = b^{\frac{r}{n}}$, and p the L. C. M. of m and n.

If the binomial is $\sqrt[m]{a^s} - \sqrt[n]{b^r}$, the factor is found in a similar manner, and is $x^{s(p-1)} + x^{s(p-2)}y^r + x^{s(p-3)}y^{2r} + \text{- - - -} + y^{r(p-1)}$.

EXAMPLES.

Ex. 1. Rationalize $\sqrt{a} + \sqrt[3]{b}$ or $a^{\frac{1}{2}} + b^{\frac{1}{3}}$.

Solution.—In this example $s=1$, $r=1$, $p=6$, $x=a^{\frac{1}{2}}$, and $y=b^{\frac{1}{3}}$. Hence formula (A) becomes $a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}}$.

This factor multiplied into $a^{\frac{1}{2}} + b^{\frac{1}{3}}$ gives $a^3 - b^2$, as the rationalized product.

2. Find a factor which will rationalize $\sqrt[3]{e^2} - \sqrt[4]{v^3}$ or $e^{\frac{2}{3}} - v^{\frac{3}{4}}$.

Result, $e^{\frac{22}{3}} + e^{\frac{20}{3}}v^{\frac{3}{4}} + e^{\frac{18}{3}}v^{\frac{6}{4}} + e^{\frac{16}{3}}v^{\frac{9}{4}} + e^{\frac{14}{3}}v^{\frac{12}{4}} + e^{\frac{12}{3}}v^{\frac{15}{4}} + e^{\frac{10}{3}}v^{\frac{18}{4}} + e^{\frac{8}{3}}v^{\frac{21}{4}} + e^{\frac{6}{3}}v^{\frac{24}{4}} + e^{\frac{4}{3}}v^{\frac{27}{4}} + e^{\frac{2}{3}}v^{\frac{30}{4}} + v^{\frac{33}{4}}$, is the required factor, and the expression rationalized is $e^8 - v^9$.

226. Prob. 8.—*A Trinomial of the form* $\sqrt{a}+\sqrt{b}+\sqrt{c}$ *may be transformed into an expression with but one radical term by multiplying it by itself with one of the signs changed, as* $\sqrt{a}+\sqrt{b}-\sqrt{c}$. *The product thus arising may then be treated as a binomial radical by considering the sum of the rational terms as one term, and the radical term as the other.* Thus, $(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})=a+b-c+2\sqrt{ab}$. Again, $[(a+b-c)+2\sqrt{ab}]\times[(a+b-c)-2\sqrt{ab}]=a^2+b^2+c^2-2ab-2bc-2ac$, a rational result.

Ex. 1. Rationalize $\sqrt{8}-\sqrt{1}-\sqrt{3}$. *Result*, 4.

SECTION IV.

COMBINATIONS OF RADICALS.

ADDITION AND SUBTRACTION.

227. Prob. 1.—To add or subtract radicals.

Rule.—*If the radicals are similar the rules already given* (**72, 77**) *are sufficient. If they are not similar make them so by* (**217-222**), *and combine as before. If they cannot be made similar, the combinations can only be indicated by connecting with the proper signs.*

Demonstration.—When the radicals are similar the radical factor is a common quantity and the coefficients show how many times it is taken. Hence the sum, or difference, of the coefficients, as the case may be, indicates how many times the common quantity is to be taken to produce the required result.

If the radicals are not similar, the reductions do not alter their values; hence the sum or difference of the reduced radicals, when they can be made similar, is the sum or difference of the radicals.

EXAMPLES.

Ex. 1. Add $\sqrt{18}$ and $\sqrt{242}$.

Model Solution.

Operation. $\sqrt{18}=3\sqrt{2}$, and $\sqrt{242}=11\sqrt{2}$.
$\therefore \sqrt{18}+\sqrt{242}=3\sqrt{2}+11\sqrt{2}=14\sqrt{2}$.

Explanation. $\sqrt{18}=\sqrt{9\times 2}$. But the square root of the product equals the product of the square roots; hence $\sqrt{9\times 2}=3\sqrt{2}$. In like manner $\sqrt{242}=\sqrt{121\times 2}=11\sqrt{2}$. Therefore $\sqrt{18}+\sqrt{242}=3\sqrt{2}+11\sqrt{2}$. But three times any quantity, as $\sqrt{2}$, and 11 times the same are 14 times that quantity. $\therefore 3\sqrt{2}+11\sqrt{2}=14\sqrt{2}$.

2. Add $\sqrt{243xy^2}$ and $\sqrt{192xy^2}$. *Sum*, $17y\sqrt{3x}$.

3. Add $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

4. Add $\sqrt{a^2y}$ and $\sqrt{c^2y}$. *Sum*, $(a+c)\sqrt{y}$.

5. From $\sqrt{605}$ take the $\sqrt{405}$. *Diff.*, $2\sqrt{5}$.

6. Add $\sqrt{605}$ and $-\sqrt{405}$.

7. Add $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

8. From $3\sqrt{\frac{2}{5}}$ take $-2\sqrt{\frac{1}{10}}$. *Diff.*, $\frac{4}{5}\sqrt{10}$.

9. What is the sum of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{1}{3}}$? *Ans.*, $\frac{5}{6}\sqrt{3}$.

10. What is the difference of $\sqrt{2ax^2-4ax+2a}$ and $\sqrt{2ax^2+4ax+2a}$? *Ans.*, $2\sqrt{2a}$.

Why should no sign be given to the last answer? If the problem read, From $\sqrt{2ax^2-4ax+2a}$ take $\sqrt{2ax^2+4ax+2a}$, why would the answer be $-2\sqrt{2a}$?

11. What is the sum of $(a-x)^2\sqrt{xy}$ and $(a+x)^2\sqrt{xy}$?

Ans., $2(a^2+x^2)\sqrt{xy}$.

12. What is the difference between $(a-x)^2\sqrt{xy}$ and $(a+x)^2\sqrt{xy}$?

Ans., $4ax^{\frac{3}{2}}y^{\frac{1}{2}}$.

13. Find the sum of $8\sqrt{\frac{3}{4}}$, $\sqrt{60}$, $-2\frac{1}{5}\sqrt{15}$, and $\sqrt{\frac{3}{5}}$.

Sum, $4\sqrt{3}$.

14. From $\sqrt[3]{\frac{27a^4x^4}{2b}}$ take $\sqrt[3]{\frac{ax^4}{54b}}$.

Rem., $\left(3ax-\frac{x}{3}\right)\sqrt[3]{\frac{ax}{2b}}$.

15. Add $a\sqrt{1+\left[\frac{b}{a}\right]^{\frac{2}{3}}}$ and $b\sqrt{1+\left[\frac{a}{b}\right]^{\frac{2}{3}}}$.

Suggestions. $a\sqrt{1+\left[\frac{b}{a}\right]^{\frac{2}{3}}}=a\sqrt{1+\frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}}=a\sqrt{\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{a^{\frac{2}{3}}}}$ $=a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}$. In like manner $b\sqrt{1+\left[\frac{a}{b}\right]^{\frac{2}{3}}}=b^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}$.

$\therefore$ The sum is $(a^{\frac{2}{3}}+b^{\frac{2}{3}})\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

16. From $(a-x)\sqrt{a^2-x^2}$ take $(a-x)\sqrt{\frac{a+x}{a-x}}$.

Rem., $(a-x-1)\sqrt{a^2-x^2}$.

17. Add $\frac{1}{x+\sqrt{x^2-1}}$ and $\frac{1}{x-\sqrt{x^2-1}}$. *Sum*, $2x$.

18. Add $\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$ and $\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

Sum, $2x^2$.

MULTIPLICATION.

***228.* Prop. 1.**—*The product of the same roots of two or more quantities, equals the like root of their product.*

Demonstration.—That is $\sqrt[m]{x}\times\sqrt[m]{y}=\sqrt[m]{xy}$. This is evident from the fact that $\sqrt[m]{xy}$ signifies that xy is to be resolved into m equal factors. If now x and y be separately resolved into m equal factors and then one factor from each be taken to make a group, there will be m such equal groups in xy. Thus $\sqrt[m]{x}$ is one of the m equal factors of x, and $\sqrt[m]{y}$ is one of the m equal factors of y. Hence $[\sqrt[m]{x}\times\sqrt[m]{y}]\cdot[\sqrt[m]{x}\times\sqrt[m]{y}]\cdot[\sqrt[m]{x}\times\sqrt[m]{y}]$ etc., to m factors of $\sqrt[m]{x}\times\sqrt[m]{y}$, makes up xy. Therefore $\sqrt[m]{x}\times\sqrt[m]{y}=\sqrt[m]{xy}$. (See Arts. **205** and **202**.)

***229.* Prop. 2.**—*Similar Radicals are multiplied by multiplying the quantities under the radical sign and writing the product under the common sign;*

Or by indicating the root by fractional indices, and, for the product, taking the common number with an index equal to the sum of the indices of the factors.

Demonstration. 1st.—Since similar radicals are the same root of the *same* quantities, as $\sqrt[m]{x}\times\sqrt[m]{x}$, this is only a particular case under **Prop. 1.**

2nd. $x^{\frac{1}{m}}\times x^{\frac{1}{m}}$ signifies that one of the m equal factors of x is to be multiplied by another of the m equal factors, or by itself. This gives 2 of the m equal factors of x, which is what is indicated by $x^{\frac{2}{m}}$.

***230.* Prob. 2.—To multiply radicals.**

Rule.—*If the factors have not the same index, reduce them to a common index, and then multiply* *the numbers under the radical sign and write the product under the common sign.*

Demonstration.—(This is the same as **Prop. 1.**)

EXAMPLES.

Ex. 1. What is the product of $\sqrt{2}$ and $\sqrt[3]{3}$?

Model Solution.

Operation. $\sqrt{2} = \sqrt[6]{8}$, and $\sqrt[3]{3} = \sqrt[6]{9}$. $\therefore\ \sqrt{2} \times \sqrt[3]{3} = \sqrt[6]{8} \times \sqrt[6]{9} = \sqrt[6]{72}$.

Explanation. $\sqrt{2} = \sqrt[6]{8}$, since the former is *one* of the *two* equal factors of 2, and the latter is *three* of the *six* equal factors of 2. In like manner $\sqrt[3]{3} = \sqrt[6]{9}$. Consequently, $\sqrt{2} \times \sqrt[3]{3} = \sqrt[6]{8} \times \sqrt[6]{9}$. Now since the product of the same root of two numbers is equal to the like root of the product, $\sqrt[6]{8} \times \sqrt[6]{9} = \sqrt[6]{72}$.

2. Multiply $\sqrt[4]{3ac}$ by $\sqrt[3]{2ac}$. *Prod.*, $\sqrt[12]{432a^7c^7}$.

3. Multiply $\sqrt{a-x}$ by $\sqrt[3]{a-x}$.
Prod., $\sqrt[6]{a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5}$.

4. Multiply $\sqrt{\frac{3}{5}}$ by $\sqrt{\frac{5}{27}}$. *Prod.*, $\frac{1}{3}$.

5. Multiply $\sqrt[5]{3}$ by $\sqrt[3]{3}$.
Prod., $3^{\frac{1}{5}+\frac{1}{3}} = 3^{\frac{8}{15}} = \sqrt[15]{6561}$.

6. Multiply $\sqrt[7]{2ax}$ by $\sqrt[5]{2ax}$.
Prod., $(2ax)^{\frac{12}{35}} = \sqrt[35]{4096a^{12}x^{12}}$.

7. Multiply $\sqrt{\frac{2}{3}}$ by $\sqrt[3]{\frac{3}{2}}$.
Prod., $\sqrt[6]{\frac{2}{3}} = \frac{1}{3}\sqrt[6]{486}$.

8. Multiply $3\sqrt{2ax}$ by $2\sqrt[3]{xy}$.

Suggestion.—Here we have the continuous product of 3, $\sqrt{2ax}$, 2, and $\sqrt[3]{xy}$. But, as the order of the factors is immaterial, we may write $3\times2\times\sqrt{2ax}\times\sqrt[3]{xy}=6\sqrt[6]{8a^3x^5y^2}$.

9. Multiply $3\sqrt[3]{\frac{2}{3}}$ by $6\sqrt[5]{\frac{1}{2}}$. *Prod.*, $6\sqrt[15]{236196}$.

10. Multiply $5a^{\frac{1}{2}}$ by $3a^{\frac{1}{3}}$.

11. Multiply $2\sqrt{ab}$ by $3\sqrt[3]{ab}$.

12. Multiply $4a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $5a^{\frac{1}{3}}b^{\frac{2}{3}}$.

13. Multiply $3x^{\frac{1}{2}}y^{\frac{1}{3}}$ by $2x^{\frac{1}{4}}y^{\frac{1}{6}}$ and represent the product without fractional exponents.

14. Multiply $\sqrt{\frac{2}{5}}$ by $\sqrt[3]{\frac{3}{8}}$ and represent the product without the use of the radical sign and in its simplest form.

Prod., $\frac{1}{10}(9000)^{\frac{1}{6}}$.

15. Multiply $a^{\frac{1}{m}}$ by $b^{\frac{1}{n}}$.

Prod., $a^{\frac{1}{m}}b^{\frac{1}{n}}$ or, $\sqrt[mn]{a^nb^m}$, or $(a^nb^m)^{\frac{1}{mn}}$.

16. Multiply $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt[4]{10}$. *Prod.*, $\frac{1}{6}\sqrt[4]{250}$.

17. Multiply $a\sqrt[m]{x}$, $b\sqrt[n]{y}$, and $c\sqrt[p]{z}$ together.

Prod., $abc\sqrt[mnp]{x^{np}y^{mp}z^{mn}}$.

18. Show that $2\sqrt[4]{9}\times16=16\sqrt{12}$.

19. Show that $\sqrt[3]{24}\times6\sqrt[6]{3}=6\sqrt{12}$.

20. Multiply $2\sqrt{a}-3\sqrt{b}$ by $3\sqrt{a}+2\sqrt{b}$.

Operation.

$$\begin{array}{lll}
2\sqrt{a} & -3\sqrt{b} & \\
\underline{3\sqrt{a}} & \underline{+2\sqrt{b}} & \\
6a & -9\sqrt{ab} & \\
 & +4\sqrt{ab} & -6b \\
\hline
6a & -5\sqrt{ab} & -6b\text{, or } 6(a-b)-5\sqrt{ab}.
\end{array}$$

21. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$. *Prod.*, $1-\sqrt{5}$.

22. Multiply $\sqrt{2}+1$ by $\sqrt{2}-1$. *Prod.*, 1.

23. Multiply $11\sqrt{2}-4\sqrt{15}$ by $\sqrt{6}+\sqrt{5}$.
Prod., $2\sqrt{3}-\sqrt{10}$.

24. Multiply $\sqrt[3]{12+\sqrt{19}}$ by $\sqrt[3]{12-\sqrt{19}}$. *Prod.*, 5.

By (**228**) $\sqrt[3]{12+\sqrt{19}}\times\sqrt[3]{12-\sqrt{19}}=\sqrt[3]{(12+\sqrt{19})(12-\sqrt{19})}$.

25. Multiply $a^2-a\sqrt{2}+1$ by $a^2+a\sqrt{2}+1$.
Prod., a^4+1.

26. Expand $(x^2+1)(x^2-x\sqrt{3}+1)(x^2+x\sqrt{3}+1)$.
Prod., x^6+1.

27. Multiply $3\sqrt{45}-7\sqrt{5}$ by $\sqrt{1\frac{4}{5}}+2\sqrt{9\frac{4}{5}}$. *Prod.*, 34.

28. Multiply $\sqrt{a}+c\sqrt[3]{b}$ by $\sqrt{a}-c\sqrt[3]{b}$.
Prod., $a-c^2\sqrt[3]{b^2}$.

DIVISION.

231. Prop.—*The quotient of the same roots of two quantities equals the like root of their quotient.*

Demonstration.—Let m be any integer and x and y any numbers; we are to prove that $\sqrt[m]{x}\div\sqrt[m]{y}$, or $\frac{\sqrt[m]{x}}{\sqrt[m]{y}}=\sqrt[m]{\frac{x}{y}}$. Now, that $\frac{\sqrt[m]{x}}{\sqrt[m]{y}}=\sqrt[m]{\frac{x}{y}}$ is evident, since $\frac{\sqrt[m]{x}}{\sqrt[m]{y}}$ raised to the mth power, that is $\frac{\sqrt[m]{x}\times\sqrt[m]{x}\times\sqrt[m]{x}\times\sqrt[m]{x}\text{ - - - to } m \text{ factors}}{\sqrt[m]{y}\times\sqrt[m]{y}\times\sqrt[m]{y}\times\sqrt[m]{y}\text{ - - - to } m \text{ factors}}=\frac{x}{y}$; whence it appears that $\frac{\sqrt[m]{x}}{\sqrt[m]{y}}$ is the mth root of $\frac{x}{y}$ or equals $\sqrt[m]{\frac{x}{y}}$. (ARTS. **206, 202.**)

232. Prob. 3.—To divide Radicals.

Rule.—*If the radicals are of the same degree, divide the number under the sign in the dividend by that under the sign in the divisor, and affect the quotient with the common radical sign.*

If the radicals are of different degrees, reduce them to the same degree before dividing.

Demonstration.—[Same as above; or, it may be considered as the converse of the corresponding case in multiplication.]

EXAMPLES.

Ex. 1. Divide $\sqrt{3a^2y^3}$ by $\sqrt[3]{2ay}$.

Model Solution.

Operation. $\sqrt{3a^2y^3} = \sqrt[6]{27a^6y^9}$, and $\sqrt[3]{2ay} = \sqrt[6]{4a^2y^2}$.

$$\therefore \frac{\sqrt{3a^2y^3}}{\sqrt[3]{2ay}} = \frac{\sqrt[6]{27a^6y^9}}{\sqrt[6]{4a^2y^2}} = \sqrt[6]{\frac{27a^6y^9}{4a^2y^2}} = \sqrt[6]{\frac{27a^4y^7}{4}} = \frac{1}{2}\sqrt[6]{432a^4y^7}.$$

Explanation.—Since $\sqrt{3a^2y^3} = \sqrt[6]{27a^6y^9}$, and $\sqrt[3]{2ay} = \sqrt[6]{4a^2y^2}$ $\frac{\sqrt{3a^2y^3}}{\sqrt[3]{2ay}} = \frac{\sqrt[6]{27a^6y^9}}{\sqrt[6]{4a^2y^2}}$. And since the quotient of the 6th root of two numbers is the 6th root of their quotient, $\frac{\sqrt[6]{27a^6y^9}}{\sqrt[6]{4a^2y^2}} = \sqrt[6]{\frac{27a^6y^9}{4a^2y^2}}$, which, by performing the operation indicated, becomes $\sqrt[6]{\frac{27a^4y^7}{4}}$; and by reducing so that the number under the radical sign shall have the integral form, this becomes $\frac{1}{2}\sqrt[6]{432a^4y^7}$.

2. Divide $\sqrt{125a^3x^2y}$ by $\sqrt{5a^2xy}$. *Quot.*, $5\sqrt{ax}$.

3. Divide $\sqrt{3}$ by $\sqrt[3]{2}$. *Quot.*, $\sqrt[6]{\frac{27}{4}} = \frac{1}{2}\sqrt[6]{432}$.

4. Divide $\sqrt{\frac{2}{3}}$ by $\sqrt[3]{\frac{1}{3}}$. *Quot.*, $\frac{1}{3}\sqrt[6]{1944}$.

5. Divide $\sqrt[3]{2a^2x^3}$ by $\sqrt[5]{2a^2x^3}$. *Quot.*, $\sqrt[15]{4a^4x^6}$.

6. Divide $\sqrt[6]{72}$ by $\sqrt{2}$. *Quot.*, $\sqrt[3]{3}$.

7. Divide $x^{\frac{2}{3}}y^{\frac{1}{3}}$ by $x^{\frac{1}{2}}y^{\frac{1}{2}}$, and represent the quotient without fractional or negative exponents. *Quot.*, $\sqrt[6]{\frac{x}{y}}$.

8. Divide $24\sqrt{ay}$ by $6\sqrt[3]{a^2y^2}$.

Suggestion. $\frac{24\sqrt{ay}}{6\sqrt[3]{a^2y^2}}=4\frac{\sqrt{ay}}{\sqrt[3]{a^2y^2}}=4\frac{\sqrt[6]{a^3y^3}}{\sqrt[6]{a^4y^4}}=4\sqrt[6]{\frac{a^3y^3}{a^4y^4}}=4\sqrt[6]{\frac{1}{ay}}$ $=4\sqrt[6]{a^{-1}y^{-1}}$.

9. Divide $125\sqrt{x^3y^3}$ by $25\sqrt[3]{x^2y^2}$, representing the quotient without the radical sign. *Quot.*, $5x^{\frac{5}{6}}y^{\frac{5}{6}}$.

10. Divide $\sqrt{6}$ by $\sqrt[8]{4}$. *Quot.*, $\sqrt{3\sqrt{2}}$.

11. Divide $20\sqrt[6]{200}$ by $4\sqrt{2}$. *Quot.*, $5\sqrt[3]{5}$.

12. Divide $\sqrt[4]{\frac{x}{y}}$ by $\sqrt[4]{\frac{y}{x}}$.

13. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$.

Suggestions. $\sqrt{2}+3\sqrt{\frac{1}{2}}=2\sqrt{\frac{1}{2}}+3\sqrt{\frac{1}{2}}=5\sqrt{\frac{1}{2}}$. Whence $\frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}}=\frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{5\sqrt{\frac{1}{2}}}=\frac{\frac{1}{2}}{5}=\frac{1}{10}$.

14. Divide $\sqrt[3]{a^2-b^2}$ by $\sqrt[3]{a-b}$. *Quot.*, $\sqrt[3]{a+b}$.

15. Divide $(a^4b^2c)^{\frac{1}{3}}$ by $(ab)^{\frac{1}{3}}$. *Quot.*, $a\sqrt[3]{bc}$.

16. Divide 200 by $\sqrt{40}$. *Quot.*, $10\sqrt{10}$, or $(10)^{\frac{3}{2}}$.

17. Divide $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$.

Suggestion.—Observe that $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}=a(\sqrt{x}+\sqrt{y})-\sqrt{b}(\sqrt{x}+\sqrt{y})=(a-\sqrt{b})(\sqrt{x}+\sqrt{y})$.

18. Divide $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

Operation.

$$\begin{array}{l|l} a+2\sqrt{ab}+b-c & \sqrt{a}+\sqrt{b}-\sqrt{c} \\ \cline{2-2} \underline{a+\sqrt{ab}-\sqrt{ac}} & \sqrt{a}+\sqrt{b}+\sqrt{c} \\ \quad\sqrt{ab}+b+\sqrt{ac}-c & \\ \quad\underline{\sqrt{ab}+b-\sqrt{bc}\qquad\quad} & \\ \qquad\qquad\sqrt{ac}+\sqrt{bc}-c & \\ \qquad\qquad\underline{\sqrt{ac}+\sqrt{bc}-c} & \end{array}$$

19. Divide $b\sqrt{a^2-b^2}$ by $a\sqrt{(a+b)^2}$.

20. Divide $\sqrt{a^2-x^2}$ by $a-x$. *Quot.*, $\sqrt{\frac{a+x}{a-x}}$.

21. Divide $m\sqrt{\frac{x-1}{x+1}}$ by $n\sqrt{\frac{x+1}{x-1}}$. *Quot.*, $\frac{m(x-1)}{n(x+1)}$.

INVOLUTION.

233. Prob. 4.—To raise a radical to any power.

Rule.—*Involve the coefficient to the required power, and also the quantity under the radical sign, writing the latter under the given sign.*

Demonstration.—This results directly from the principles of multiplication of radicals (**230**). Thus, to raise $a\sqrt[n]{b}$ to the mth power, is to take m factors of $a\sqrt[n]{b}$, which gives $a\sqrt[n]{b}\times a\sqrt[n]{b}\times a\sqrt[n]{b}$, etc., to m factors. But as the order of the factors is immaterial (**85**) this may be written aaa - - - to m factors $\times\ \sqrt[n]{b}\times\sqrt[n]{b}\times\sqrt[n]{b}$ to m factors. But aaa - - - to m factors is by notation a^m, and $\sqrt[n]{b}\times\sqrt[n]{b}\times\sqrt[n]{b}$ - - - to m factors is by (**230**) $\sqrt[n]{b^m}$. $\therefore$ The mth power of $a\sqrt[n]{b}$ is $a^m\sqrt[n]{b^m}$. Q. E. D.

EXAMPLES.

Ex. 1. Raise $\frac{1}{3}\sqrt{\frac{2}{5}}$ to the 3rd power.

Model Solution.

Operation. $(\frac{1}{3}\sqrt{\frac{2}{5}})^3 = \frac{1}{3}\sqrt{\frac{2}{5}} \cdot \frac{1}{3}\sqrt{\frac{2}{5}} \cdot \frac{1}{3}\sqrt{\frac{2}{5}} = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \times \sqrt{\frac{2}{5}} \cdot \sqrt{\frac{2}{5}} \cdot \sqrt{\frac{2}{5}} = \frac{1}{27}\sqrt{\frac{8}{125}}$ or $\frac{2}{135}\sqrt{\frac{2}{5}} = \frac{2}{675}\sqrt{10}$.

Explanation.—The cube of $\frac{1}{3}\sqrt{\frac{2}{5}}$ is a product three factors each $= \frac{1}{3}\sqrt{\frac{2}{5}}$; but as the order of the factors is immaterial this may be considered as 3 factors each $= \frac{1}{3}$, or $\frac{1}{27}$, and 3 factors each $= \sqrt{\frac{2}{5}}$, or $\frac{2}{5}\sqrt{\frac{2}{5}}$. Hence $(\frac{1}{3}\sqrt{\frac{2}{5}})^3 = \frac{1}{27} \times \frac{2}{5}\sqrt{\frac{2}{5}} = \frac{2}{135}\sqrt{\frac{2}{5}}$. Which by removing the denominator from under the radical sign becomes $\frac{2}{675}\sqrt{10}$.

2. Raise $2\sqrt[3]{3a^2b}$ to the second power.

3. Raise $\frac{1}{2}\sqrt[3]{4x^2y}$ to the 5th power.

4. Raise $-3\sqrt{\frac{2}{7}}$ to the 3rd power.

5. Square $3-\sqrt{2}$. *Square*, $11-6\sqrt{2}$.

6. Cube $\sqrt{2}-\sqrt{3}$. *Cube*, $11\sqrt{2}-9\sqrt{3}$.

7. Cube $2\sqrt{x-y}$.

Cube, $8(x-y)\sqrt{x-y}$, or $8(x-y)^{\frac{3}{2}}$.

234. **Cor.**—*To raise a radical to a power whose index is the index of the root, is simply to drop the radical sign.* Thus, the square of $\sqrt{ab}$ is ab, the cube of $\sqrt[3]{2x^2y}$ is $2x^2y$, the square of $\sqrt{-2a^3b}$ is $-2a^3b$, the 5th power of $\sqrt[5]{a^2-b^2}$ is a^2-b^2.

Scholium.—This process of involution is a special case under (**194**).

EVOLUTION.

235. **Prob. 5.—To extract any required root of a monomial radical.**

Rule.—*Extract the root of the coefficient, and of the quantity under the radical sign separately, affecting* *the latter with the given radical sign. Reduce the result to its simplest form.*

Demonstration.—The nth root of $a\sqrt[m]{b}$, signifies one of the n equal factors of $a\sqrt[m]{b}$. Hence if we resolve a into n equal factors, and $\sqrt[m]{b}$, into n equal factors, a group consisting of one from each is the nth root of $a\sqrt[m]{b}$. Thus one of the n equal factors of a is $\sqrt[n]{a}$, and of $\sqrt[m]{b}$, $\sqrt[m]{\sqrt[n]{b}}$, for $\sqrt[m]{\sqrt[n]{b}} \times \sqrt[m]{\sqrt[n]{b}} \times \sqrt[m]{\sqrt[n]{b}}$ - - - to n factors, is $\sqrt[m]{\sqrt[n]{b \cdot b \cdot b \cdot b \text{ to } n \text{ factors}}} = \sqrt[m]{b}$ (**233, 234**). If now we take a group consisting of one from each set of factors, that is $\sqrt[n]{a}\sqrt[m]{\sqrt[n]{b}}$, we have the nth root of $a\sqrt[m]{b}$, since we have one of its n equal factors. Q. E. D.

EXAMPLES.

Ex. **1.** What is the 3rd root of $4\sqrt{3a^3x}$?

Model Solution.

Operation. $\sqrt[3]{4\sqrt{3a^3x}} = \sqrt[3]{4}\sqrt[3]{\sqrt{3a^3x}} = \sqrt[3]{4}\sqrt[6]{3a^3x}$ $= \sqrt[6]{16}\sqrt[6]{3a^3x} = \sqrt[6]{48a^3x}$.

Explanation.—The cube root of $4\sqrt{3a^3x}$ is one of the 3 equal factors which compose it. In order to find this, I resolve each of the two factors 4 and $\sqrt{3a^3x}$ into 3 equal factors, and take one of each. 4 resolved into 3 equal factors becomes $\sqrt[3]{4} \times \sqrt[3]{4} \times \sqrt[3]{4}$. (A process which in reality is only indicated.) In like manner $\sqrt{3a^3x}$, resolved into 3 equal factors becomes $\sqrt{\sqrt[3]{3a^3x} \times \sqrt[3]{3a^3x} \times \sqrt[3]{3a^3x}}$ or $\sqrt{\sqrt[3]{3a^3x}} \times \sqrt{\sqrt[3]{3a^3x}} \times \sqrt{\sqrt[3]{3a^3x}}$ since the root of the product equals the product of the roots. Now taking one factor out of each of these groups I have $\sqrt[3]{4}\sqrt{\sqrt[3]{3a^3x}}$, and as three such factors could be formed from the number, $\sqrt[3]{4}\sqrt{\sqrt[3]{3a^3x}}$ is the cube root of $4\sqrt{3a^3x}$. But this expression can be reduced to a more simple form, by observing first that the square root of the cube root is the 6th root, and then reducing $\sqrt[3]{4}$ to the same radical form. Thus I have $\sqrt[3]{4}\sqrt{\sqrt[3]{3a^3x}}$ $= \sqrt[3]{4}\sqrt[6]{3a^3x} = \sqrt[6]{16}\sqrt[6]{3a^3x} = \sqrt[6]{48a^3x}$ by (**233**).

2. Extract the cube root of $\sqrt{a^4x^6}$. *Root*, $x\sqrt[3]{a^2}$.

3. Extract the square root of $32\sqrt[3]{192a^8x^2}$.

Suggestion.—The square root of $32\sqrt[3]{192a^8x^2} = 4\sqrt{2} \times \sqrt[6]{192a^8x^2}$ $= 4\sqrt{2} \times 2a\sqrt[6]{3a^2x^2} = 8a\sqrt{2}\sqrt[6]{3a^2x^2} = 8a\sqrt[6]{8}\sqrt[6]{3a^2x^2} = 8a\sqrt[6]{24a^2x^2}$.

4. Extract the cube root of $\frac{1}{8}a^3\sqrt{b}$.

5. Extract the 4th root of $16a^2\sqrt{x}$. *Root*, $2\sqrt[8]{a^4x}$.

6. Extract the cube root of $(a+x)\sqrt{a+x}$.

Root, $\sqrt{a+x}$.

7. Extract the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$. *Root*, $\frac{1}{3}\sqrt{3a}$.

8. Extract the square root of $\frac{1}{2}\sqrt{\frac{1}{2}}$. *Root*, $\frac{1}{2}\sqrt[4]{2}$.

236. **Scholium.**—This operation is but a special case of *affecting a quantity with any given exponent* (**194**), and the examples may be performed according to the rule there given. Thus, to extract the cube root of $8\sqrt{3ax^2}$; putting it in the form $8 \times (3ax^2)^{\frac{1}{2}}$ and dividing exponents by 3 (multiplying by $\frac{1}{3}$) we have $8^{\frac{1}{3}}(3ax^2)^{\frac{1}{6}} = 2\sqrt[6]{3ax^2}$. The pupil may solve the following in this manner:

9. Extract the 5th root of $\sqrt{32x^{10}}$.

Suggestion. $\sqrt{32x^{10}} = (32)^{\frac{1}{2}}x^5$. Multiplying exponents by $\frac{1}{5}$ we have $(32)^{\frac{1}{10}}x$. But $(32)^{\frac{1}{10}} = \sqrt{(32)^{\frac{1}{5}}} = \sqrt{2}$. $\therefore \sqrt[5]{\sqrt{32x^{10}}} = x\sqrt{2}$. But the most simple way to solve this particular case is $\sqrt[5]{\sqrt{32x^{10}}}$ $= \sqrt{\sqrt[5]{32x^{10}}} = \sqrt{2x^2} = x\sqrt{2}$. Or by the rule given (**235**).

10. Extract the square root of $\sqrt[3]{49a^2}$.

11. Extract the cube root of $64\sqrt[4]{8a^6}$.

12. Extract the 5th root of $486a\sqrt[3]{4a^2}$. *Root*, $3\sqrt[3]{2a}$.

FOR REVIEW OR ADVANCED COURSE.

237. **Prob. 6.—To extract the square root of a binomial, one or both of whose terms are radicals of the second degree.**

Solution.—Such binomials have either the form $a \pm n\sqrt{b}$ or $m\sqrt{a} \pm n\sqrt{b}$. Now observing that $(x \pm y)^2 = x^2 \pm 2xy + y^2$, we see that if we can separate the first term of any such binomial surd into two parts the square root of the product of which shall be $\frac{1}{2}$ the other term, these two parts may be made the first and third terms of a trinomial (corresponding to $x^2 + 2xy + y^2$), and the middle term being the second term of the given binomial, the square root will be the sum or difference of the square roots of the parts into which the first term is separated.

EXAMPLES.

Ex. **1.** Extract the square root of $87 - 12\sqrt{42}$.

Solution.—Let x = one of the parts into which 87 is to be separated, and $87 - x$ the other. Then we have $\sqrt{x(87-x)} = -6\sqrt{42}$, or squaring, $87x - x^2 = 1512$, or $x^2 - 87x = -1512$. $\therefore$ $x = 63$ or 24, and we have $\sqrt{87 - 12\sqrt{42}} = \sqrt{63 - 12\sqrt{42} + 24}$. Now $\sqrt{63} = 3\sqrt{7}$ and $\sqrt{24} = 2\sqrt{6}$, and as the middle term of the trinomial is negative and twice the product of these roots, its square root is $3\sqrt{7} - 2\sqrt{6}$. **(123.)**

2. Extract the square root of $3\sqrt{6} + 2\sqrt{12}$.

Suggestion. $3\sqrt{6}$ is to be separated into two parts. Let them be x and $3\sqrt{6} - x$. Then $x(3\sqrt{6} - x) = 12$. Whence $x = 2\sqrt{6}$ or $\sqrt{6}$, and $\sqrt{3\sqrt{6} + 2\sqrt{12}} = \sqrt{2\sqrt{6} + 2\sqrt{12} + \sqrt{6}} = \sqrt{2\sqrt{6}} + \sqrt{\sqrt{6}}$ $= \sqrt[4]{24} + \sqrt[4]{6}$.

3. Extract the square root of $12-\sqrt{140}$.

Root, $\sqrt{7}-\sqrt{5}$.

4. Extract the square root of $11+6\sqrt{2}$.

5. Extract the square root of $13+2\sqrt{30}$.

6. Extract the square root of $ax-2a\sqrt{ax-a^2}$.

Suggestion.—Letting y be one of the parts into which ax is to be separated, the equation from which its value is found is $y^2-axy = -a^2(ax-a^2)$ or $-a^3x+x^4$. Whence $y=ax-a^2$ or a^2, and the parts are $ax-a^2$ and a^2. Hence $\sqrt{ax-2a\sqrt{ax-a^2}}$ $=\sqrt{(ax-a^2)-2a\sqrt{ax-a^2}+a^2}=\sqrt{ax-a^2}-\sqrt{a^2}$ or $a-\sqrt{ax-a^2}$.

7. Extract the square root of $2a+2\sqrt{a^2-b^2}$.

IMAGINARY QUANTITIES.

238. **An Imaginary Quantity** is an *indicated even* root of a *negative* quantity, or any expression, taken as a whole, which contains such a form either as a factor or a term. Thus $\sqrt[6]{-b}$, $\sqrt{-a^2}$, $5\sqrt{-b^2}$, $2+\sqrt{-4}$, $\sqrt[4]{-6}$, $3-\sqrt[6]{-1}$, etc., are imaginary quantities.

239. **Scholium 1.**—It is a mistake to suppose that such expressions are in any proper sense more unreal than other symbols. The term Impossible Quantities should not be applied to them: it conveys a wrong impression. The limits of this work prevent anything more than a mere explanation of the method of multiplying or dividing one imaginary of the second degree by another.

240. **Scholium 2.**—A curious property of these symbols, and one which for some time puzzled mathematicians, appears when we attempt to multiply $\sqrt{-a}$ by $\sqrt{-a}$. Now the square root of any quantity multiplied by itself, should, by definition, be the quantity itself; hence $\sqrt{-a} \times \sqrt{-a} = -a$. But if we apply the process of multiplying the quantities under the radicals we have $\sqrt{-a} \times \sqrt{-a} = \sqrt{a^2} = +a$, as well as $-a$. What then is the product of $\sqrt{-a} \times \sqrt{-a}$? Is it $-a$, or is it both $+a$ and $-a$? The true product is $-a$; and the explanation is, that $\sqrt{a^2}$ is, *in general*, $+a$ or $-a$. But when we know what factors were multiplied together to produce a^2, and the nature of our discussion limits us to these, the sign of $\sqrt{a^2}$ is no longer ambiguous; it is the same as was its root.

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241. **Prop.**—*Every imaginary term of the second degree (and in fact of every other degree) can be reduced to the form* $m\sqrt{-1}$ *in which* m *is not imaginary.* m *may be rational or surd.*

Demonstration.—Let $\sqrt{-x}$ represent any such expression. Then $\sqrt{-x} = \sqrt{x(-1)} = \sqrt{x}\sqrt{-1}$, which is the required form.

242. **Scholium 3.**—By means of this proposition and the property noticed in *Sch.* 2, the multiplication and division of imaginaries is effected.

EXAMPLES.

Ex. 1. Multiply $4\sqrt{-3}$ by $2\sqrt{-2}$.

Operation. $4\sqrt{-3} = 4\sqrt{3} \times \sqrt{-1}$. Also $2\sqrt{-2} = 2\sqrt{2} \times \sqrt{-1}$. Hence $4\sqrt{-3} \times 2\sqrt{-2} = 2 \times 4\sqrt{3} \times \sqrt{2} \times \sqrt{-1} \times \sqrt{-1} = 8\sqrt{6} \times \sqrt{-1} \times \sqrt{-1} = -8\sqrt{6}$, since $\sqrt{-1} \times \sqrt{-1} = -1$.

2. Multiply $3\sqrt{-5}$ by $4\sqrt{-3}$. *Prod.*, $-12\sqrt{15}$.

3. Multiply $\sqrt{-x^2}$ by $\sqrt{-y^2}$. *Prod.*, $-xy$.

4. Multiply $2\sqrt{-9}$ by $3\sqrt{-4}$. *Prod.*, -36.

5. Multiply $2\sqrt{-6}$, $5\sqrt{-4}$, $3\sqrt{-7}$ together.
Prod., $-60\sqrt{42}\sqrt{-1}$, or $-60\sqrt{-42}$.

6. Multiply $24+\sqrt{-49}$ by $24-\sqrt{-49}$.

Suggestion.—We have here the product of the sum and difference of two quantities which equals the difference of their squares. Hence $(24+\sqrt{-49})\times(24-\sqrt{-49})=(24)^2-(\sqrt{-49})^2=576-(-49)=576+49=625$.

7. Divide $\sqrt{-16}$ by $\sqrt{-4}$.

Operation. $\sqrt{-16}=4\sqrt{-1}$, and $\sqrt{-4}=2\sqrt{-1}$. Hence $\dfrac{\sqrt{-16}}{\sqrt{-4}}=\dfrac{4\sqrt{-1}}{2\sqrt{-1}}=2$.

Scholium.—A superficial view of the case might lead one to think that the quotient was ±2. Thus, noticing that the radicals are similar, he might conclude that $\dfrac{\sqrt{-16}}{\sqrt{-4}}=\sqrt{\dfrac{-16}{-4}}=\sqrt{4}=\pm 2$, an incorrect result.

8. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Quot.*, $\frac{3}{2}\sqrt{3}$.

9. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$. *Quot.*, $\frac{1}{18}\sqrt{3}$.

10. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

Operation.—Writing the example thus $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}\cdot$ and multiplying both terms of the fraction by $1+\sqrt{-1}$ there results, (see *Ex.* 6) $\dfrac{1+2\sqrt{-1}-1}{1-(-1)}=\dfrac{2\sqrt{-1}}{2}=\sqrt{-1}$.

The example may also be performed thus:

$$\begin{array}{c|c} 1+\sqrt{-1} & -\sqrt{-1}+1 \\ \cline{2-2} 1+\sqrt{-1} & \sqrt{-1} \end{array}$$

Since 1 divided by $-\sqrt{-1}$ gives $\sqrt{-1}$, as $-\sqrt{-1}\times\sqrt{-1}=1$.

SYNOPSIS FOR REVIEW.

- **POWERS AND ROOTS.**
  - **DEFINITIONS.**
    - **Power.—Degree of.** / **Root.—Degree of.**
      - SCH. 1.—Powers and Roots correlative.
      - SCH. 2.—Square, Cube, etc.
    - **Exponent or Index.**
      - 1. + INTEGER.
      - 2. + FRACTION.
      - 3. − INT. OR FR.
      - *Cor.* Transferring a factor in a fraction from one term to another.
    - **Radical.**
      - REAL.
        - Rational.
        - Irrational.
      - IMAGINARY.
      - HOW INDICATED.
        - $\sqrt{\ }$
        - $\frac{m}{n}$
    - **Sim. Rads.** RATIONALIZATION. To affect with Exponents.
    - **Involution.** EVOLUTION. Calculus of radicals.
  - **INVOLUTION.**
    - **Prob. 1.** To any power. RULE. *Dem.* *Cor.*—Signs.
    - **Prob. 2.** To affect with Exponents. RULE. *Dem.*
      - 1. $+m$.
      - 2. $+\frac{m}{n}$.
      - 3. $-m$ or $-\frac{m}{n}$.
    - **Prob. 3.** BINOMIAL FORMULA.
      - *Cor.* 1.—Terminates.
      - *Cor.* 2.—Number of Terms.
      - *Cor.* 3.—Coefficients.
      - *Cor.* 4.—Exponents in ea. term.
      - *Cor.* 5.—RULE.
      - *Cor.* 6.—$(a-b)^m$.
  - **EVOLUTION.**
    - **Prob. 1.** Roots of Perfect Powers. RULE. *Dem.*
      - *Sch.*—Signs.
      - *Cor.* 1.—Same as (193).
      - *Cor.* 2.—R. of Prod. = Prod. of Roots.
      - *Cor.* 3.—R. of Quot. = Quot of Roots.
    - **Prob. 2.** Square Root of Poly. RULE. *Dem.*
      - *Sch.* 1.
      - *Sch.* 2.
      - *Sch.* 3.
    - **Prob. 3.** Square Root of Dec. No. RULE. *Dem.*
      - *Sch.* 1.
      - *Sch.* 2.
    - **Prob. 4.** Cube Root of Poly. RULE. *Dem.*
      - *Sch.* 1.
      - *Sch.* 2.
    - **Prob. 5.** Cube Root of Dec. No. RULE. *Dem.*
      - *Sch.* 1.
      - *Sch.* 2.
      - *Sch.* 3.
    - **Prob. 6.** Any Root whose index composed of factors, 2 or 3.
    - **Prob. 7.** Any Root. Solution. General Scholium.

SYNOPSIS FOR REVIEW.—*Continued.*

- CALCULUS OF RADICALS.
  - REDUCTIONS.
    - **Prob. 1.** Remove factor. RULE. *Dem.*
      - *Sch.* Simplest form.
      - *Cor.* Denomination of surd.
    - **Prob. 2.** Index composite, etc. RULE. *Dem.*
    - **Prob. 3.** To given index. RULE. *Dem.*
    - **Prob. 4.** To common index. RULE. *Dem.*
    - **Prob. 5.** To rationalize Monomial Denominator. RULE. *Dem.*
    - **Prob. 6.** To rationalize Binomial Denominator. RULE. *Dem.*
    - **Prop. 1.** To rationalize any binomial. *Dem.*
    - **Prop. 2.** To rationalize any trinomial. *Dem.*
  - COMBINATIONS.
    - **Prob. 1.** To add and subtract. RULE. *Dem.*
    - **Multip.**
      - **Prop. 1.** Product of roots = equal root of product. *Dem.*
      - **Prop. 2.** Similar Radicals, how multiplied. *Dem.*
      - **Prob. 2.** To multiply Radicals. RULE. *Dem.*
    - **Div.**
      - **Prop.** Quotient of roots = root of quotients.
      - **Prob. 3.** To divide Radicals. RULE. *Dem.*
    - **Prob. 4.** Involution of Radicals. RULE. *Dem.* *Sch.*
    - **Prob. 5.** Evolution of Radicals. RULE. *Dem.*
    - **Prob. 6.** $\sqrt{a+\sqrt{b}}$ and $\sqrt{\sqrt{a}+\sqrt{b}}$, etc.
  - IMAGINARIES.
    - **Definitions.**
      - *Sch.* 1.—Called Impossible.
      - *Sch.* 2.—Reason for name.
    - **Prob. 1.** To add or subtract. RULE. *Dem.*
    - **Prob. 2.** To multiply. RULE. *Dem.*
    - **Prob. 3.** To divide. RULE. *Dem.*

Test Questions.—By what must numerator and denominator of $\dfrac{\left(1+\dfrac{x^2}{y^2}\right)^{\frac{3}{2}}}{\dfrac{x^2}{y^2}}$ be multiplied to reduce it to a simple fraction? Give the various significations of an exponent. Perform the operation $\sqrt{2}\times\sqrt[3]{3}$, and explain the process. Repeat the Binomial Formula, and by means of it expand $(1-x^2)^{\frac{1}{2}}$. Demonstrate the rules for square and cube root. Show that $\dfrac{a+bx-\sqrt{a^2+b^2x^2}}{a-bx+\sqrt{a^2+b^2x^2}}=\dfrac{\sqrt{a^2+b^2x^2}-a}{bx}$. What sign is given to a square root? Why? To a cube root? Why? What is the value of x^0?

[**Note.**—Here ends the subject of Literal Arithmetic. The student is now prepared for the study of Algebra, properly so called; *i. e., The Science of the Equation.*]

PART II.

ALGEBRA.

CHAPTER I.

Simple Equations

SECTION I.

EQUATIONS WITH ONE UNKNOWN QUANTITY.

DEFINITIONS.

1. **An Equation*** is an expression in mathematical symbols, of equality between two numbers or sets of numbers.

Illustration. $3x - 2a^2y = \frac{5-3y^2}{2}$ is an equation because it is an expression of equality between $3x-2a^2y$ and $\frac{5-3y^2}{2}$.

2. **Algebra** is that branch of Pure Mathematics which treats of the nature and properties of the Equation and of its use as an instrument for conducting mathematical investigations.

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* Do not pronounce this word "Equazion." For this common error there is no authority. "Equashun" is the correct pronunciation.

3. **The First Member** of an equation is the part on the left hand of the sign of equality. **The Second Member** is the part on the right.

4. **A Numerical Equation** is one in which the *known* quantities are represented by decimal numbers; as $12x^2 - 3x = 48$.

5. **A Literal Equation** is one in which some or all of the known quantities are represented by letters; as $ax - c + 3by = \frac{4cx^2 - 2}{3}$.

6. **The Degree** of an Equation is determined by the highest number of unknown factors occurring in any term.

Illustration. $ax - bx^2 = c + x^3$, is of the 3rd degree; $a^2x - 4x = 12$ is of the 1st degree; $x^2y^2 = 18$ is of the 4th degree, etc.

7. **A Simple Equation** is an equation of the first degree.

Illustration. $y = ax + b$ is a simple equation, as also is $\frac{x-3}{2} + 4x = \frac{1}{3}x + 5$.

8. **A Quadratic Equation** is an equation of the second degree.

9. **A Cubic Equation** is an equation of the third degree. **A Biquadratic** is one of the fourth degree.

10. Equations above the second degree are called **Higher Equations.**

TRANSFORMATION OF EQUATIONS.

11. **To Transform** an Equation is to change its form without destroying the equality of the members.

12. There are *four* principal transformations of simple Equations containing one unknown quantity, viz.: *Clearing of Fractions*, *Transposition*, *Collecting Terms*, and *Dividing* by the coefficient of the unknown quantity.

13. These transformations are based upon the following

AXIOMS.

AXIOM 1.—*Any operation which does not affect the value of a term or member, may be performed upon that term or member without destroying the Equation.*

AXIOM 2.—*If both members of an Equation are increased or diminished alike, the equality is not destroyed.*

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14. Prob.—To clear an Equation of Fractions.

Rule.—*Multiply each member by the least or lowest common multiple of all the denominators.*

Demonstration.—This process clears the Equation of fractions, since, in the process of multiplying any particular fractional term, its denominator is one of the factors of the L. C. M. by which we are multiplying; hence dropping the denominator multiplies by this factor, and then this product (the numerator) is multiplied by the other factor of the L. C. M.

This process does not destroy the Equation, since both members are increased or diminished alike.

EXAMPLES.

Ex. 1. Clear the equation $\frac{x}{2}+\frac{x}{3}+\frac{x}{6}=\frac{2x+3}{3}$ of fractions.

Model Solution. 6 is the L. C. M. of 2, 3, 6, and 3, the denominators. Now, it is evident that, if the first member of this equation is equal to the second, 6 times the first member is equal to 6 times the second; hence I can multiply each member by 6 and not destroy the equality.

Therefore taking the equation

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{6}=\frac{2x+3}{3},$$

and multiplying each member by 6, I have

$$3x+2x+x=4x+6.$$

The operation is performed thus: 2 times $\frac{x}{2}$ is x; and 3 times x is $3x$. 3 times $\frac{x}{3}$ is x; and 2 times x is $2x$. 6 times $\frac{x}{6}$ is x. Hence 6 times the first member is $3x+2x+x$.

3 times $\frac{2x+3}{3}$ is $2x+3$; and 2 times $2x+3$ is $4x+6$. Hence 6 times the second member is $4x+6$.

Thus the denominators have all been caused to disappear without destroying the equality of the members of the equation, as both have been increased alike.

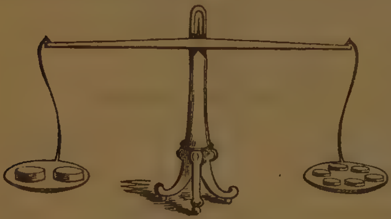

Illustration.—An equation is aptly compared to a pair of scales with equal arms, balanced by weights in the two pans.

Now, if the weights in the scale pans balance each other, that is, *are equal*, and we multiply the weights in each pan by 6 (or any other number), the balance (equality) will still be preserved. Or, if we increase or decrease the weights in both pans *equally*, the balance (equality) will not be destroyed.

2. Clear $\frac{2x}{3}-\frac{5x}{4}+1=\frac{11}{2}$ of fractions.

Result, $8x-15x+12=66$.

3. Clear $10+\frac{2x-5}{5}=\frac{1+x}{2}-3x$ of fractions.

Result, $100+4x-10=5+5x-30x$.

4. Clear $\frac{x-1}{2}-\frac{x-2}{3}+x=\frac{3x-1}{6}+\frac{4-x}{3}$ of fractions.

Suggestion.—The multiplier is 6. In multiplying the second term, $\frac{x-2}{3}$, it should be borne in mind that the $-$ sign preceding this compound term, shows that the term *as a whole* is to be subtracted. Hence when this term is written without any mark of aggregation, its signs are to be changed, as in removing terms from a bracket preceded by a minus sign. The equation cleared of fractions is $3x-3-2x+4+6x=3x-1+8-2x$. Why are not the signs changed in the last term, $\frac{4-x}{3}$? Are all the signs changed in the term $\frac{x-2}{3}$? Yes. What becomes of the $-$ sign before this fraction in the given example? It is dropped after the operations signified by it have been expressed in detail. We might write the equation cleared of fractions thus: $3x-3-(2x-4)+6x=3x-1+8-2x$, the term $2x-4$ being still taken in the aggregate. Now removing the parenthesis (give the reason) we have $3x-3-2x+4+6x=3x-1+8-2x$, as above.

NEGLECT TO MAKE THIS CHANGE OF SIGNS IS ONE OF THE MOST COMMON MISTAKES OF BEGINNERS.

5. Clear $\frac{x}{m}+\frac{x}{am}-\frac{x}{m^2}=4c-\frac{3x-n}{a^2m}$ of fractions.

Result, $a^2mx+amx-a^2x=4a^2cm^2-3mx+mn$.

6. Clear $\frac{x}{4}-\frac{x-3}{2}=\frac{a}{6}$ of fractions.

Result, $3x-6x+18=2a$.

7. Clear $\frac{2x}{3a}-\frac{x-1}{2ab}+\frac{3}{a^2}=1-x$ of fractions.

Result, $4abx-3ax+3a+18b=6a^2b-6a^2bx$.

8. Clear $\frac{x}{a-b}-x+\frac{3}{a+b}=\frac{a-b}{a+b}-1$ of fractions.

Suggestion.—The multiplier is a^2-b^2.

Result, $ax+bx-a^2x+b^2x+3a-3b=(a-b)^2-a^2+b^2$.

9. Clear $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$ of fractions.

Result, $ax+2ac-bx-2bc=ax+bx-ac-bc$.

10. Clear $\frac{2x}{a-2b}=3+\frac{x}{2a-b}$ of fractions.

Result, $4ax-2bx=6a^2-15ab+6b^2+ax-2bx$.

TRANSPOSITION.

15. **Transposing** a term is changing it from one member of the equation to the other without destroying the equality of the members.

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16. Prob.—To transpose a term.

Rule.—*Drop it from the member in which it stands and insert it in the other member with its sign changed.*

Demonstration.—If the term to be transposed is +, dropping it from one member diminishes that member by the amount of the term, and writing it with the – sign in the other member, takes its amount from that member; hence both members are diminished alike, and the equality is not destroyed. (Repeat AXIOM 2.)

2nd.—If the term to be transposed is –, dropping it *increases* the member from which it is dropped, and writing it in the other member with the + sign *increases* that member by the same amount; and hence the equality is preserved. (Repeat AXIOM 2.)

EXAMPLES.

Ex. 1. Given the equation $3+2x-5=12-4x$ to transpose so that all the terms containing the unknown quantity, x, shall stand in the first member and the known terms in the second member.

Model Solution.

Operation. $3+2x-5=12-4x$

$2x+4x=12-3+5$

Explanation.—Dropping 3 from the first member diminishes that member by 3; hence to preserve the equality, I subtract 3 from the second member, *or indicate* the subtraction by writing it in the second member with the $-$ sign. Thus the term 3 is transposed.

Dropping -5 from the first member *increases* that member by 5; and hence to preserve the equality I add 5 to the second member. Thus the term -5 is transposed.

Dropping $-4x$ from the second member *increases* that member by $4x$; hence I increase the first member by adding $4x$ to it, and thus preserve the equality.

I have thus arranged the terms so that all those containing the unknown quantity stand in the first member, and all known terms in the second member; and yet I have preserved the equality.

Illustration.—This operation can be illustrated by the scales on page 214. Suppose the positive terms to represent *weights* and the negative terms some forces *lifting* on the scale-pans. [Since the $+$ and $-$ terms represent quantities opposed in effects.] Taking the 3 from the first member corresponds to taking off so much weight. This can be compensated, so as to keep the scales in equilibrium, by applying a lifting power of 3 to the other side, which is symbolized by -3 on that side. Again taking -5 from the first member is like taking away a *lifting* power of 5, which can be compensated by putting a *weight* of 5 on the other side $(+5)$. In like manner the transposition of any term can be *illustrated. In fact all operations upon equations can be illustrated in a similar way.*

In the following examples transpose the unknown terms to the first member and the known to the second.

2. Given $5x-12a+3c-2x=4x-2x+4a$ to transpose as above.

Result, $5x-2x-4x+2x=4a+12a-3c.$

3. Given $100+4x-6=5x+5-30x$ to transpose the terms as above.

Result, $4x-5x+30x=5-100+6.$

4. Transpose as above $\frac{15}{2}-\frac{3x}{2}+10-\frac{7}{8}=\frac{1}{2}-\frac{11}{5}x+x.$

Result, $\frac{11}{5}x-x-\frac{3x}{2}=\frac{1}{2}+\frac{7}{8}-10-\frac{15}{2}.$

5. Transpose as above $3ab-\frac{2a^2}{c}-4ax+18bx^2=\frac{2ax}{5}-x^2$.

$$Result,\ 18bx^2+x^2-4ax-\frac{2ax}{5}=\frac{2a^2}{c}-3ab.$$

SOLUTION OF SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

17. To **Solve** an Equation is to find the value of the unknown quantity: that is, to find what value it must have in order that the equation be true.

Illustration.—In the equation $4x-2=2x+4$, if we call x, 3, the first member is 10, and as the second is also 10 for this value of x, the equation is true. But if we try any other number than 3 for x, we shall find that the equation will not be true. Thus trying 4 for x, we find the first member 14 and the second 12; and the equation is not true. Again, try 5. The first member becomes 18 and the second 14, and the equation is not true.

Let the student see if he can ascertain by *inspection* what are the values of x in the following:

$$x+3=3x+1.$$
$$2x=30-x.$$

Though these equations are very simple, it is probable that it will take the student some time to *guess* out the values of x which make them true.

But, if it is so difficult to hit upon just the value of x which is required to make so simple an equation true, the task would be quite hopeless in such an one as

$$\frac{3x-1}{5}-\frac{13-x}{2}=\frac{7x}{3}-\frac{11(x+3)}{6}.$$

Yet we have a very simple method of solving any such equation so as to tell certainly and easily what the value of x is. This process is now to be explained, and is called *Solving the Equation*, or sometimes the *Resolution of the Equation*.

18. An equation is said to be **Satisfied** when a value is given to the unknown quantity which makes it a true equation: *i. e.*, which makes its members equal. It is said to be **Destroyed** when its members by any process become unequal.

19. To **Verify** an equation is to substitute the supposed value of the unknown quantity and thus see if it satisfies the equation.

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20. Prob. 1.—To solve a simple equation containing one unknown quantity.

Rule.—*I. If the equation contain fractions, clear it of them by* **Art 14.**

II. Transpose all the terms involving the unknown quantity to the first member, and the known terms to the second member by **Art. 15.**

III. Unite all the terms containing the unknown quantity into one by addition, and put the second member into its simplest form.

IV. Divide each member by the coefficient of the unknown quantity.

Demonstration.—The first step, clearing of fractions, does not destroy the equation, since each member is multiplied by the same quantity (AXIOM 2).

The second step does not destroy the equation, since it is adding the same quantity to each member, or subtracting the same quantity from each member (AXIOM 2).

The third step does not destroy the equation, since it does not change the value of the members (AXIOM 1).

The fourth step does not destroy the equation, since it is dividing each member by the same quantity, and thus changes the members alike (AXIOM 2).

Hence, after these several processes, we still have a true equation. But now the first member is simply the unknown quantity, and the second member is all known. Thus we have *what the unknown quantity is equal to; i. e., its value.*

21. Scholium I.—*It must be fixed in the pupil's mind that he can make but two classes of changes upon an equation without destroying it:* viz., SUCH AS DO NOT AFFECT THE VALUE OF THE MEMBERS, or SUCH AS AFFECT BOTH MEMBERS EQUALLY. *Every operation must be seen to conform to these conditions.*

EXAMPLES.

Ex. 1. Solve $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$, and verify the result.

Model Solution.

Operation.

(1) - - - - $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$,

(2) - - - - $20x+20+24x-32+5=30x+35$,

(3) - - - - $20x+24x-30x=35-20+32-5$,

(4) - - - - $14x=42$,

(5) - - - - $x=3$.

Explanation.—I first clear equation (1) of fractions by multiplying each member by the L. C. M. of its denominators, which is 40. This does not destroy the equation (AXIOM 2). I multiply $\frac{x+1}{2}$ by 40 by dropping its denominator 2, thus getting $x+1$, as the result of multiplying by 2, and then multiply $x+1$ by 20, getting $20x+20$. [In like manner explain the entire process of clearing of fractions.]

Having cleared the equation of fractions I have (2). I now transpose the terms containing x to the first member and the known terms to the second member. Thus, dropping $30x$ from the second member diminishes it by that amount, whence to preserve the equality of the members I subtract $30x$ from the first member, *i. e.*, write it in that member with its sign changed. [In like manner explain the transposition of each term.]

I know equation (3) to be true, since I have changed both members alike, that is, have added to and subtracted from each member the same quantities. I now add together the terms of the first member, *which does not affect the value of the member*, and have $14x$. In the same manner uniting the terms of the second member does not alter its value; hence $14x=42$. Finally, I divide each member by 14 and have $x=3$. This operation does not destroy the equation, since *each* member of the equality $14x=42$ is divided by the same number (AXIOM 2). Hence 3 is the value of x.

Verification.

I will now see by actual trial if 3 does satisfy the given equation. Substituting 3 for x, I have

$$\frac{3+1}{2}+\frac{3\times 3-4}{5}+\frac{1}{8}=\frac{6\times 3+7}{8}, \text{ or}$$

$$\frac{4}{2}+\frac{5}{5}+\frac{1}{8}=\frac{25}{8}, \text{ or}$$

$$2+1+\frac{1}{8}=3\tfrac{1}{8}, \text{ or}$$

$$3+\frac{1}{8}=3\tfrac{1}{8},$$

the members of which are identical; and the value of x is verified.

2. Solve and verify $\frac{2x+1}{2}=\frac{7x+5}{8}$. *Result*, $x=1$.

Verification. $\frac{2+1}{2}=\frac{7+5}{8}$, or $\frac{3}{2}=\frac{12}{8}$. Whence it appears that 1 satisfies the equation.

3. Solve and verify $2x-\frac{1}{2}=\frac{3x-1}{2}+\frac{1+x}{3}$.

Result, $x=2$.

4. Given $\frac{2(x-1)}{3}-\frac{9}{5}=\frac{x+3}{7}-\frac{4}{5}$ to find x.

Result, $x=4$.

5. Given $7x+6-3x=56+2x$ to find x, and verify.

6. Given $\frac{y-5}{4}+6y=\frac{284-y}{5}$ to find y.

7. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find x.

Result, $x=9$.

8. Given $\frac{5(x-5)}{20}+6x=\frac{284-x}{5}$ to find x.

9. Given $25(11x-8)=18(12x+2)$ to find x.

10. Given $\frac{3x}{5}+2\frac{1}{2}+11=\frac{x}{4}+17$ to find x.

Result, $x=10$.

11. Given $y+\frac{13-y}{3}=\frac{17-y}{2}$ to find y.

Result, $y=3\frac{4}{7}$.

Verification. $3\frac{4}{7}+\frac{13-3\frac{4}{7}}{3}=\frac{17-3\frac{4}{7}}{2}$, or

$$3\frac{4}{7}+\frac{9\frac{3}{7}}{3}=\frac{13\frac{3}{7}}{2}, \text{ or}$$

$$\frac{10\frac{5}{7}+9\frac{3}{7}}{3}=6\frac{5}{7}, \text{ or}$$

$$\frac{20\frac{1}{7}}{3}=6\frac{5}{7},$$

which is a true equation, since $20\frac{1}{7}\div 3$ is $6\frac{5}{7}$.

In verifying it is not well to go through the processes of clearing of fractions, transposition, etc., but rather keep the terms as distinct as possible, and reduce each member separately to a form so simple that they can be seen to be equal.

12. Given $\frac{z-2}{3}+3z=\frac{5+z}{2}+1$, to resolve and verify as above.

Result, $z=1\frac{8}{17}$.

Verification. $\frac{\frac{25}{17}-2}{3}+\frac{75}{17}=\frac{5+\frac{25}{17}}{2}+1$, which reduces to $-\frac{3}{17}+\frac{75}{17}=\frac{55}{17}+\frac{17}{17}$, or $\frac{72}{17}=\frac{72}{17}$.

13. Given $3y+\frac{1-y}{5}-3=\frac{3-10y}{4}-2\frac{1}{2}$ to resolve and verify as above.

Result, $y=\frac{21}{106}$.

14. Given $\frac{3(x-1)}{5}+\frac{2(x-2)}{15}=\frac{2(2-x)}{3}$.

Result, $x=\frac{11}{7}$.

15. Find the value of x in $3x + \frac{2x+6}{5} = 5 + \frac{11x-37}{2}$.

Suggestion.—Having cleared this equation of fractions, transposed the unknown terms to the first member, and the known to the second, there results $-21x = -147$. Hence to obtain a *positive* result divide each member by -21, and $x = 7$.

22. Cor. 1.—*All the signs of the terms of both members of an equation can be changed from + to —, or* vice versa, *without destroying the equality, since this is equivalent to multiplying or dividing each member by* -1.

16. Find the value of x in $x + \frac{18-4x}{3} = \frac{6x-4}{3} - 2$.

Result, $x = 4$.

17. Given $\frac{97-7y}{2} + \frac{5(y-1)}{8} = 21 + \frac{3y-11}{16}$ to find y.

Result, $y = 9$.

18. Find the value of z in $1 + \frac{5+z}{2} = 3z + \frac{z-2}{3}$. (See *Ex.* 12.)

Result, $z = \frac{25}{17}$.

19. Find the value of x in $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{x+3}{4}$.

Suggestion.—In clearing of fractions be careful to notice the term $-\frac{x+3}{4}$. As this term is to be subtracted, when the sign of aggregation (the line between the terms of the fraction) is dropped, the signs of the separate terms must be changed. The equation when cleared is $6x+6+4x+8 = 192-3x-9$; and $x = 13$. It is well for the pupil to explain such cases thus: Having multiplied $\frac{x+3}{4}$ by the L. C. D. 12, I have $3x+9$; but this is to be subtracted, hence I annex it with its signs changed, as subtraction is performed by - - - -.

20. What value for z satisfies $z+\frac{3z-5}{2}=12-\frac{2z-4}{3}$?

21. Find the value of z which satisfies the equation $3z-\frac{z-4}{4}-4=\frac{5z+14}{3}-\frac{1}{12}$.

22. Given $\frac{3x+7}{14}-\frac{2x-7}{21}=\frac{x-4}{4}-2\frac{3}{4}$, to find x.

Separating the fractions into parts, and reducing, $\frac{3}{14}x+\frac{1}{2}-\frac{2}{21}x+\frac{1}{3}=\frac{1}{4}x-1-2\frac{3}{4}$. Whence $(\frac{3}{14}-\frac{2}{21}-\frac{1}{4})x=-\frac{1}{2}-\frac{1}{3}-3\frac{3}{4}$, or $-\frac{11}{84}x=-\frac{55}{12}$, or $\frac{1}{7}x=5$, and $x=35$.

23. Given $\frac{1}{3}(7x+9)+\frac{1}{14}(249-9x)=\frac{1}{4}(9x-13)+\frac{1}{7}(3x+1)$, to find x.

24. Solve $\frac{1}{2}(x+1)+\frac{1}{3}(x+2)=16-\frac{1}{4}(x+3)$.

25. Solve $\frac{1}{4}(3x-3)-\frac{1}{3}(3x-4)=5\frac{1}{3}-\frac{1}{9}(27+4x)$.

26. Solve $\frac{1}{6}(8-x)+x-1\frac{2}{3}=\frac{1}{2}(x+6)-\frac{1}{3}x$.

27. Given $\frac{3a+x}{a}-5x=6$, to find x.

Operation.—Multiplying each member by a we have $3a+x-5ax=6a$. Transposing, $x-5ax=3a$, or $(1-5a)x=3a$.

[**Note.**—It is the common experience of pupils that they continue to find difficulty with *Literal Equations* even after they are quite familiar with *Numerical* ones, such as the 26 above given. *This difficulty* MUST *be overcome.* No one has caught the true spirit of mathematical reasoning till the literal notation is seen and *felt* to be more simple than the decimal.]

28. What value for x satisfies $ax+b=cx+d$? Verify it.

Ans., $x=\frac{d-b}{a-c}$.

Verification.—Substituting the value of x for x we have $a\frac{d-b}{a-c}+b=c\frac{d-b}{a-c}+d$. Performing the multiplications indicated, and reducing each member to an improper fraction, this becomes, $\frac{ad-ab+ab-bc}{a-c}=\frac{cd-bc+ad-cd}{a-c}$. Now $-ab$ and $+ab$ destroy each other in the first member, and $+cd$ and $-cd$ in the second. Hence we have $\frac{ad-bc}{a-c}=\frac{ad-bc}{a-c}$, which is evidently true.

29. Given $ax+b^2=bx+a^2$ to find the value of x.

Result, $x=a+b$.

30. Given $ax+m=bx+n$ to find x. *Result*, $x=\frac{n-m}{a-b}$.

31. Given $bx+2x-a=3x-2c$ to find x.

Result, $x=\frac{a-2c}{b-1}$.

32. What value for x satisfies $ax+b=\frac{x}{a}+\frac{1}{b}$?

Ans., $x=\frac{a(1-b^2)}{b(a^2-1)}$.

33. Solve $\frac{1-cy}{m}=\frac{1-my}{c}$ and verify the result.

34. Find x from $ax-\frac{a^2-3bx}{a}-ab^2=bx+\frac{6bx-5a^2}{2a}-\frac{bx+4a}{4}$.

Result, $x=\frac{4ab^2-10a}{4a-3b}$.

35. Solve $\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc}$.

Result, $x=\frac{a^2c+ab^2+bc^2-a-b-c}{ac+ab+bc-1}$.

36. Given $\frac{1}{a}+\frac{b}{x}=c$ to find x. *Result*, $x=\frac{ab}{ac-1}$.

37. Given $\frac{x-a}{3}-\frac{2x-3b}{5}-\frac{a-x}{2}=10a+11b$ to find x.

Result, $x=25a+24b$.

38. Given $\frac{ab}{x}-bc=m+\frac{1}{x}$ to find x.

Result, $x=\frac{ab-1}{bc+m}$.

39. Given $\frac{x}{a}-1+3ab=\frac{dx}{c}$ to find x.

Result, $x=\frac{ac(1-3ab)}{c-ad}$.

40. What is the value of x in $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$.

23. **Scholium 3.**—It is not always expedient to perform the several transformations *in the same order* as given in the rule. The process may often be much shortened by the exercise of a little ingenuity. The ultimate object is to so transform the equation that the unknown quantity will stand alone in the first member.

An expeditious method of solving the last example is as follows: Multiplying by 9, we have $6x+7+\frac{21x-39}{2x+1}=6x+12$. (The term $\frac{7x-13}{6x+3}$ is multiplied by 3 by dividing the denominator, and by the other factor of 9, by multiplying the numerator.) Now dropping $6x$ from each member and subtracting 7, we get $\frac{21x-39}{2x+1}=5$. Whence $x=4$.

41. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$ to find x.

Suggestion.—Multiply numerator and denominator of the term $\frac{4x+3}{9}$ by 2, transpose and unite it to the second member, and there results $\frac{7x-29}{5x-12}=\frac{13}{18}$. Whence $x=6$.

42. Given $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{10\frac{1}{4}}{14}$ to find x.

43. Given $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}$ to find x.

44. Given $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}$ to find x, and verify the result.

Result, $x=-2$.

Verification. $\frac{-12+1}{15}-\frac{-4-4}{-14-16}=\frac{-4-1}{5}$, or $-\frac{11}{15}-\frac{4}{15}=-1$, or $-\frac{15}{15}=-1$.

45. Given $\frac{2(4x+3)}{x+3}+\frac{3}{x+1}=8$ to find x, and verify the value.

46. Solve for x the equation $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}$.

An elegant solution of this is obtained by first uniting the two terms of the first member, which makes the equation $\frac{bx-ax+ax}{a(b-a)}=\frac{a}{b+a}$, or $\frac{b}{a(b-a)}x=\frac{a}{b+a}$. Whence $x=\frac{a^2(b-a)}{b(b+a)}$.

47. Solve $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0$.

Suggestion.—Performing the multiplications and transposing, we have $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=\frac{a}{6}-\frac{a}{12}+\frac{a}{20}$, or $\frac{5}{12}x=\frac{2}{15}a$. Whence dividing by $\frac{5}{12}$ we have $x=\frac{8}{25}a$.

48. Solve $\frac{ay}{b}-\frac{c}{d}=\frac{b}{a}-\frac{dy}{c}$.

Result, $y=\frac{bc}{ad}$.

49. Solve $\frac{2x}{a-2b} = 3 + \frac{x}{2a-b}$.

Result, $x = 2a-5b+\frac{2b^2}{a}$.

Suggestion.—Transpose $\frac{x}{2a-b}$ to the first member and unite the terms, giving $\frac{3a}{2a^2-5ab+2b^2}x = 3$. $\therefore\ x = 2a-5b+\frac{2b^2}{a}$.

50. Solve $3x - a = x - \frac{bx-d}{3}$.

51 Solve $\frac{x(a-b)}{2} = a + \frac{ab}{4} - \frac{x}{3}$.

Suggestion. $\frac{ax-bx}{2}+\frac{x}{3}=a+\frac{ab}{4}$, or $\frac{3a-3b+2}{6}x=\frac{4a+ab}{4}$. $\therefore$ $x=\frac{4a+ab}{4}\times\frac{6}{3a-3b+2}=\frac{3a(4+b)}{6(a-b)+4}$.

52. Solve $\frac{3y-a}{b}+\frac{y+2b}{c}=\frac{7y}{c}-\frac{a}{4}$.

Result, $y=\frac{8b^2-4ac+abc}{12(2b-c)}$.

53. Solve $\frac{1}{3}(x-a)-\frac{1}{4}(2x-3b)-\frac{1}{2}(a-x)=b-\frac{5}{6}a$.

Result, $x=\frac{3}{4}b$.

54. If $\frac{(a-b)y}{2}-\frac{(a+b)y}{3}+3(ab-2by)=3ab$, show that $y=0$.

55. Solve $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$. *Result,* $x=\frac{b}{c}$.

Suggestion.—First divide each member by a, and then write the first member in two parts, reducing each fraction to its lowest terms. Drop $\frac{x}{b}$ from each member and $\frac{b}{x}=c$.

56. Solve $\frac{(2x+3)x}{2x+1}+\frac{1}{3x}=x+1$.

Suggestion.—An elegant solution of this is obtained by noticing that $\frac{(2x+3)x}{2x+1}=\frac{2x^2+3x}{2x+1}=x+1-\frac{1}{2x+1}$; whence the equation becomes $x+1-\frac{1}{2x+1}+\frac{1}{3x}=x+1$. Dropping $x+1$ from each member, transposing and changing signs, $\frac{1}{2x+1}=\frac{1}{3x}$, or $3x=2x+1$. $\therefore\ x=1$.

57. Solve $\frac{1}{2}+\frac{2}{x+2}=\frac{x+2}{2x}$.

Suggestion. $\frac{x+2}{2x}=\frac{1}{2}+\frac{1}{x}$. $\therefore\ \frac{2}{x+2}=\frac{1}{x}$, and $x=2$.

58. Solve $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$, and verify.

Suggestion.—First destroy the term $\frac{2x}{5}$.

59. Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

Suggestion.—Reducing each term to a mixed number we have $1+\frac{1}{x-2}-1-\frac{1}{x-3}=1+\frac{1}{x-6}-1-\frac{1}{x-7}$. Whence $\frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-6}-\frac{1}{x-7}$. Reducing the terms in each member separately to common denominators and adding, we get $\frac{1}{(x-2)(x-3)}=\frac{1}{(x-6)(x-7)}$. Whence $(x-2)(x-3)=(x-6)(x-7)$, or $x^2-13x+42=x^2-5x+6$, and $x=4\frac{1}{2}$.

24. Scholium 4.—It often happens that an equation which involves the second or even higher powers of the unknown quantity is still, virtually, a simple equation, since these terms destroy each other in the reduction.

60. What value for x satisfies $3-x-2(x-1)(x+2) = (x-3)(5-2x)$?

Ans., $x = 1\frac{4}{7}$.

61. Solve the equation $(x-5)(x-2)-(x-5)(2x-5)+(x+7)(x-2)=0$?

Ans., $x = 2\frac{3}{13}$.

62. Solve $\frac{x+10}{3} - \frac{3}{5}(3x-4) + \frac{(3x-2)(2x-3)}{6} = x^2 - \frac{8}{15}$, and verify.

63. Solve $(x+\frac{5}{2})(x-\frac{3}{2}) - (x+5)(x-3) + \frac{3}{4} = 0$, and verify the result.

64. Solve $(a+x)(b+x) = (c+x)(d+x)$.

Result, $x = \frac{cd-ab}{a+b-c-d}$.

65. Solve $\frac{a+b}{x-c} = \frac{a}{x-a} + \frac{b}{x-b}$.

Result, $x = \frac{ab(a+b-2c)}{a^2+b^2-ac-bc}$.

66. Solve $(a+x)(b+x)-a(b+c) = \frac{a^2c}{b} + x^2$.

Suggestion.—Perform the multiplication indicated in the first member, and write the terms *without* clearing of fractions; this gives $(a+b)x - ac = \frac{a^2c}{b}$. Whence $(a+b)x = \frac{a^2c+abc}{b} = \frac{(a+b)ac}{b}$. Dividing by $(a+b)$ we have $x = \frac{ac}{b}$.

SIMPLE EQUATIONS CONTAINING RADICALS.

25. Many equations containing radicals which involve the unknown quantity, though not primarily appearing as simple equations, become so after being freed of such radicals.

26. Prob. 2.—To free an equation of Radicals.

Rule.—*I. Transpose the terms so that the radical, if there is but one, or the more complex radical, if there are several, shall constitute one member.*

II. Involve each member of the equation to a power of the same degree as the radical.

III. If a radical still remains repeat the process, being careful to keep the members in the most condensed form and lowest terms.

Demonstration.—That this process frees the equation of the radical which constitutes one of its members is evident from the fact that a radical quantity is involved to a power of the same degree as its indicated root by dropping the root sign.

That the process does not destroy the equality of the members is evident from the fact that the like powers of equal quantities are equal. Both members are increased or decreased alike.

EXAMPLES.

Ex. 1. Find the value of x in $\sqrt{4x+16}=12$.

Model Solution.

Operation.

$$\begin{aligned}\sqrt{4x+16}&=12.\\ 4x+16&=144.\\ 4x&=128.\\ x&=32.\end{aligned}$$

Explanation.—I first square each member of the equation. The first member, $\sqrt{4x+16}$, is squared by dropping its radical sign, since the square of a square root is the number itself. The square of the second member, 12, is 144. This process is equivalent to multiplying the first member by $\sqrt{4x+16}$ and the second by 12, hence as $\sqrt{4x+16}$ is equal to 12, both members have been increased alike. [The equation being freed from radicals the explanation becomes the same as before.]

2. Solve $\sqrt{5x+6}=4$, and verify.

3. Solve $\sqrt{10x+3} = 7$. *Result,* $x = \frac{23}{5}$.

4. Solve $\sqrt{2x+3}+4 = 7$, and verify.

Suggestion.—First transpose the 4 and unite it with the 7; otherwise, squaring will not free the equation of radicals.

5. Solve $8+\sqrt{3x+6} = 14$, and verify.

6. Solve and verify $3\sqrt{2x+6}+3 = 15$.

Result, $x = 5$.

7. Solve $\sqrt{ax+2ab}-a = b$. *Result,* $x = \frac{a^2+b^2}{a}$.

8. Solve $\sqrt{x+x^2}-x-\frac{1}{3} = 0$. *Result,* $x = \frac{1}{3}$.

9. Solve $ax+a\sqrt{2ax+x^2} = ab$. *Result,* $x = \frac{b^2}{2(a+b)}$.

Suggestion.—Before squaring put the equation in the form $\sqrt{2ax+x} = b-x$.

10. Given $\sqrt{12+y}-\sqrt{y} = 2$ to find the value of y.

Result, $y = 4$.

Queries.—If each member is squared as the equation stands, will the equation be freed from radicals? Is the first member a binomial or a trinomial? What is its square? Which will give the most simple result, to square it as it stands or to transpose one of the radicals? Which one is it best to transpose? Will once squaring free it from radicals?

11. Given $\sqrt{x-16} = 8-\sqrt{x}$ to solve and verify.

Suggestion.—Solve this and the five following like the preceding by squaring twice.

12. Solve $\sqrt{a+x}+\sqrt{a-x}=\sqrt{ax}$.

Result, $x=\dfrac{4a^2}{a^2+4}$.

13. Solve $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$.

Result, $x=\dfrac{25a}{16}$.

14. Solve $\sqrt{5}\times\sqrt{x+2}=\sqrt{5x}+2$.

Result, $x=\dfrac{9}{20}$.

15. Solve $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$.

Result, $x=\dfrac{b^2-4a^2}{4a}$.

16. Solve $2\sqrt{b+x}-\sqrt{4a+x}=\sqrt{x}$.

Result, $x=\dfrac{(b-a)^2}{2a-b}$.

Query.—Why is it best to transpose one of the terms of the first member to the second before squaring?

17. Given $x+\sqrt{a-x}=\dfrac{a}{\sqrt{a-x}}$ to find x.

Scholium I.—It is frequently best to clear the equation of fractions first, even when it involves radicals, especially if the denominator is of such a form as will not make the equation complicated.

In this case, clearing of fractions we have $x\sqrt{a-x}+a-x=a$. Whence $x\sqrt{a-x}=x$, or $\sqrt{a-x}=1$. Finally $x=a-1$.

18. Solve $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$.

Suggestion.—The pupil should exercise his ingenuity in seeing in what different ways he can solve such examples, and notice the most elegant methods. For example, compare the following:

1*st Method.*—Clearing of fractions $x^2-ax^2=x$. Dividing by x, $x-ax=1$, $\therefore\ x=\dfrac{1}{1-a}$.

2nd Method.—Multiplying each member by $\sqrt{x}$ we have $x-ax=1$ as before.

3rd Method.—Dividing numerator and denominator of the second member by $\sqrt{x}$, we have $\frac{x-ax}{\sqrt{x}}=\frac{1}{\sqrt{x}}$. Whence multiplying each member by $\sqrt{x}$, $x-ax=1$, etc.

4th Method.—Divide the numerator of the first member by its denominator and $\sqrt{x}-a\sqrt{x}=\frac{\sqrt{x}}{x}$. Dividing each member by $\sqrt{x}$, $1-a=\frac{1}{x}$, or $x=\frac{1}{1-a}$.

19. Solve $\frac{\sqrt{ax}-b}{\sqrt{ax}+b}=\frac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$. *Result,* $\frac{9b^2}{a}$.

Suggestion.—The more elegant solution of this is to reduce each member to a mixed number, obtaining $1-\frac{2b}{\sqrt{ax}+b}=1-\frac{7b}{3\sqrt{ax}+5b}$. Dropping the 1, and dividing by $-b$, $\frac{2}{\sqrt{ax}+b}=\frac{7}{3\sqrt{ax}+5b}$. Clearing of fractions $6\sqrt{ax}+10b=7\sqrt{ax}+7b$. $\therefore\ \sqrt{ax}=3b$, etc.

In a similar manner solve the two following:

20. Solve $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$. *Result,* $x=4$.

21. Solve $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}$. *Result,* $x=\left(\frac{ab}{a-b}\right)^2$.

22. Solve $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}$. *Result,* $x=3$.

Suggestion. $\frac{3x-1}{\sqrt{3x}+1}=\sqrt{3x}-1$.

23. Solve $\frac{x-4}{\sqrt{x}+2}=5\sqrt{x}-8+\frac{3\sqrt{x}}{2}$.

Suggestion.—In this and the following use the same expedient as in the 22nd. Thus $\sqrt{x}-2=5\sqrt{x}-8+\frac{3}{2}\sqrt{x}$, or $5\frac{1}{2}\sqrt{x}=6$, or $\sqrt{x}=\frac{12}{11}$. $\therefore x=\frac{144}{121}$.

24. Solve $\dfrac{x-a}{\sqrt{x}+\sqrt{a}}=\dfrac{\sqrt{x}-\sqrt{a}}{3}+2\sqrt{a}$.

Suggestion.—We have $\sqrt{x}-\sqrt{a}=\frac{1}{3}\sqrt{x}+1\frac{2}{3}\sqrt{a}$, or $\frac{2}{3}\sqrt{x}=2\frac{2}{3}\sqrt{a}$. $\therefore x=16a$.

25. Solve $\dfrac{ax-b^2}{\sqrt{ax}+b}=c+\dfrac{\sqrt{ax}-b}{c}$.

Result, $x=\dfrac{1}{a}\left(b+\dfrac{c^2}{c-1}\right)^2$.

26. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\sqrt{m}$. *Result*, $x=\dfrac{2a\sqrt{m}}{1+m}$.

Suggestion.—Rationalize the denominator (**224**), and after reducing $\sqrt{a^2-x^2}=x\sqrt{m}-a$.

27. Solve $\dfrac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=c$. *Result*, $x=\dfrac{a(c-1)^2}{4c}$.

28. Solve $\dfrac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}-2\sqrt{x}}=9$. *Result*, $x=\dfrac{4}{9}$.

29. Solve $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\sqrt{\dfrac{x}{x+\sqrt{x}}}$.

Suggestion.—Multiply each member by $\sqrt{x+\sqrt{x}}$, and after reduction $\sqrt{x^2-x}=x-\frac{1}{2}\sqrt{x}$. Squaring, transposing, and combining, $\sqrt{x^3}=\frac{5}{4}x$. $\therefore x=\frac{25}{16}$.

30. Solve $\sqrt{\sqrt{x}+3}-\sqrt{\sqrt{x}-3}=\sqrt{2\sqrt{x}}$.

Result, $x=9$.

31. Solve $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$.

Result, $x=\frac{a(b+c)}{b-c}$.

32. Solve $\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}$. $x=2a$.

33. Solve $.15x+1.575-.875x=.0625x$. $x=2$.

34. Solve $1.2x-\frac{.18x-.05}{.5}=.4x+8.9$. $x=20$.

35. Solve $4.8x-\frac{.72x-.05}{.5}=1.6x+8.9$. $x=5$.

36. Solve $2.3x-\frac{x-.25x}{.4}=3.2-\frac{x}{3}$.

37. Solve $x-.24=\frac{3.4-.6x}{.21}$. $x=4.259+$.

SUMMARY OF PRACTICAL SUGGESTIONS.

27.—In attempting to solve a simple equation which does not contain radicals, consider,

1. Whether it is best to clear of fractions first.

2. Look out for compound negative terms.

3. If the numerators are polynomials and the denominators monomials, it is often better to separate the fractions into parts. (*Ex.* 22, p. 224 is an illustration.)

4. It is often expedient, when some of the denominators are monomial or simple, and others polynomial or more complex, to clear of the most simple first, and after each step see that by transposition, uniting terms, etc., the equation is kept in as simple a form as possible. (See *Exs.* 40 to 44.)

5. It is sometimes best to transpose and unite some of the terms before clearing of fractions. (See *Exs.* 46 to 54.)

6. Be constantly on the lookout for a factor which can be canceled from each member of the equation (*Ex.* 55), or for terms which destroy each other (*Exs.* 60 to 66).

7. It sometimes happens that by reducing fractions to mixed numbers the terms will unite or destroy each other, especially when there are several polynomial denominators. (See *Ex.* 59.)

28. WHEN THE EQUATION CONTAINS RADICALS, CONSIDER,

1. If there is but *one* radical, by causing it to constitute one member and the rational terms the other, the equation can be freed by involving each member to the power denoted by the index of the radical. (See *Exs.* 1 to 9.)

2. If there are two radicals and other terms, make the more complex radical constitute one member, alone, before squaring. Such cases usually require two involutions. (See *Exs.* 10 to 16.)

3. If there is a radical denominator and *radicals of a similar form* in the numerators or constituting other terms, it may be best to clear of fractions first, either in whole or part. (See *Exs.* 17, 18, 29.)

4. Frequently a fraction may be reduced to a whole or mixed number with advantage. (See *Exs.* 19 to 25.)

5. It is sometimes best to rationalize a radical denominator. (See *Exs.* 26 to 28.)

6. Always watch for an opportunity to cancel a factor, or drop terms which destroy each other.

APPLICATIONS.

29. According to the definition (2), Algebra treats of, 1st, The nature and properties of the Equation; and 2nd, the method of using it as an instrument for mathematical investigation.

The Simple Equation has been so thoroughly discussed in the preceding part of the section, that the pupil will now be able to use it in solving problems.

30. **The Algebraic Solution** of a problem consists of two parts:

1st. **The Statement,** which consists in expressing by one or more equations the conditions of the problem.

2nd. **The Solution** of these equations so as to find the values of the unknown quantities in known ones. This process has been explained, in the case of Simple Equations, in the preceding articles.

31. The *Statement* of a problem requires some knowledge of the subject about which the question is asked. Often it requires a great deal of this kind of knowledge in order to "state a problem." This is not Algebra; but it is knowledge which it is more or less important to have according to the nature of the subject.

32. **Directions** to guide the student in the *Statement of Problems:*

1st. Study the meaning of the problem, so that, *if you had the answer given, you could prove it*, noticing carefully just what operations you would have to perform upon the answer in proving. This is called, *Discovering the relations between the quantities involved.*

2nd. Represent the unknown (required) quantities (the answer) by some one or more of the final letters of the alphabet, as x, y, z, or w, and the known quantities by the other letters, or numbers, given in the problem.

3rd. Lastly, by combining the quantities involved, *both known and unknown*, according to the conditions given in the problem (as you would to prove it, if the answer were known) express these relations in the form of an equation.

PROBLEMS.

Ex. 1. What number is that to the double of which if 18 be added, the sum is 82?

Model Solution.

Statement.—Let x represent the unknown number sought. Then the problem says that double this number, that is $2x$, with 18 added, that is $2x+18$, is 82. Hence $2x+18=82$ is the *statement.*

Resolution of the Equation.—[With this the pupil is familiar.] Solving $2x+18=82$, we find $x=32$. 32 is, therefore, the number sought.

Verification. $2\times32+18=82$.

[**Note.**—In this example there are three members involved, the 18, 82, and the one to be found, which we call x. The relations between these numbers are explicitly stated in the problem, and the statement is easily made. This is not always so. These relations are often the most difficult thing to discover, and their discovery requires a knowledge of other subjects than Algebra. Suppose the problem to be: Given three masses of metal of equal weight, one of pure gold, one of pure silver, and one part gold and part silver. When they are immersed in water, the gold displaces 5 ounces, the silver 9 ounces, and the compound 6 ounces. What part of the last was gold and what part silver? Now in this problem the relations between the quantities are not explicitly stated. Yet by a knowledge of *Natural Philosophy* and a little of *Mensuration*, they can be found out, and the statement of the problem made in an equation.]

2. What number is that, to the double of which if 44 be added, the sum is 4 times the required number?

Suggestion.—Suppose I guess that the number sought is 30, how will you tell whether I am right or not? You say: "Double 30 and add 44 to it, and, if you are right, the sum will equal 4 times 30." But $2\times30+44$ does *not* $=4\times30$; so I am wrong. Now, call the number sought x, and use it as 30 was used in proving that it is *not* the answer, and the statement, $2x+44=4x$ is obtained. Whence $x=22$. Verify it.

3. What number is that, the double of which exceeds its half by 6? *Ans.*, 4.

4. The property of two persons is \$16000, and one owns three times as much as the other. How much has each?

Statement. Let $x =$ the less amount.
Then $3x =$ the greater amount.
And $3x+x = 16000$.
$\therefore\ x = 4000$ and $3x = 12000$.

5. A man being asked his age replied: "If to my age you add its half, and third, and then deduct 10, the result is 100." What was his age? *Ans.*, 60.

6. After paying $\frac{1}{7}$ of a bill and $\frac{1}{5}$ of it, \$92 still remained due. What was the bill at first?

Statement.—Let $x =$ the amount of the bill; then $\frac{1}{7}x$ and $\frac{1}{5}x$ had been paid. Taking these amounts from the bill we have $x-\frac{1}{7}x-\frac{1}{5}x$. But this, by the problem, was \$92. Hence $x-\frac{1}{7}x-\frac{1}{5}x = 92$. $\therefore\ x = 140$.

7. The sum of 2 numbers is 180 and the difference 10. What are the numbers?

Statement.—Let $x =$ the less number: then $x+10$ is the greater, and $x+x+10 = 180$. $\therefore\ x = 85$, and $x+10 = 95$, and the two numbers are 85 and 95.

8. The sum of two numbers is 5760, and their difference is $\frac{1}{3}$ the greater. What are they?

Statement.—Let $x =$ the greater; then $5760-x$ is the less, and $x-(5760-x) = \frac{x}{3}$. $\therefore\ x = 3456$, and the less is 2304.

9. A man divided 80 cents among four beggars; to the first two he gave equal amounts, to the third twice as much as to one of these, and to the fourth twice as much as to the third. How much did he give to each?

The equation is $x+x+2x+4x = 80$. Whence he gave the first and second 10c. each, the third 20c., and the fourth 40c.

10. A, B, C, and D, invest \$4755 in a speculation. B furnishes 3 times as much as A, C as much as A and B together, and D as much as C and B. How much does each invest?

Ans., A, \$317; B, \$951; C, \$1268; and D, \$2219.

THE SAME PROBLEMS WITH THE LITERAL NOTATION.

11. What number is that, to n times which if m be added, the sum is s?

The equation is $nx+m=s$. *Ans.*, $\frac{s-m}{n}$.

Scholium.—To make this conform to PROB. 1, we call $s=82$, $m=18$, and $n=2$. $\therefore \frac{s-m}{n}=\frac{82-18}{2}=32$. But the answer $\frac{s-m}{n}$ applies just as well to any other problem of the kind, no matter what numbers are involved. Thus, let the problem be,—What number is that to 5 times which if 20 be added, the sum is 100? Now $s=100$, $m=20$, and $n=5$. Whence $\frac{s-m}{n}=\frac{100-20}{5}=16$. *We, therefore, see that* $\frac{s-m}{n}$ *is a general answer to all such problems.*

12. What number is that, to a times which if b be added, the sum is c times the number?

Ans., Equation, $ax+b=cx$. $\therefore$ The number is $\frac{b}{c-a}$.

Queries.—How is this adapted to PROB. 2? What other problems can you state to which this value of x affords an answer?

13. What number is that, a times which exceeds $\frac{1}{b}$ times it by m?

Equation, $ax-\frac{x}{b}=m$. $\therefore x=\frac{bm}{ab-1}$ is the number.

Queries.—How is this adapted to PROB. 3? What other problems can you state to which $\frac{bm}{ab-1}$ affords an answer? Repeat the same inquiries after each of the seven following examples.

14. The property of two persons is \$$m$, and one owns n times as much as the other. How much has each?

Ans., $\frac{m}{1+n}$ and $\frac{mn}{1+n}$.

15. What number is that to which if $\frac{1}{m}$th and $\frac{1}{n}$th of itself be added, and then a deducted, the result is b?

Ans., $\frac{(a+b)mn}{mn+m+n}$.

16. After paying $\frac{1}{m}$th and $\frac{1}{n}$th of a bill, a remained due. What was the bill?

Ans., $\frac{amn}{mn-n-m}$.

17. The sum of two numbers is s, and their difference d. What are the numbers?

Ans., $\frac{s+d}{2}$ and $\frac{s-d}{2}$.

33. **Cor.**—*Observe that the solution of this problem proves the very useful theorem:*

Half the sum plus half the difference of two quantities is the greater of the quantities, and half the sum minus half the difference is the less.

Thus, the sum of two numbers is 20 and their difference 12. What are the numbers? $\frac{s}{2}+\frac{d}{2}=10+6=16$, the greater; and $\frac{s}{2}-\frac{d}{2}=10-6=4$, the less.

18. The sum of two numbers is s and their difference is $\frac{1}{m}$th of the greater. What are the numbers.

Equation, $x-(s-x)=\frac{x}{m}$. *Ans.*, $\frac{ms}{2m-1}$, $\frac{s(m-1)}{2m-1}$.

19. A man divided \$$a$ among 4 beggars; to the first two he gave equal amounts, to the third m times as much as to each of these, and to the fourth m times as much as to the third. How much did he give to each?

Ans., To each of the first two \$$\frac{a}{2+m+m^2}$, to the third \$$\frac{am}{2+m+m^2}$, and to the fourth \$$\frac{am^2}{2+m+m^2}$.

20. A, B, C, and D invest \$$s$ in a speculation. B furnishes m times as much as A; C as much as A and B together, and D as much as C and B together. How much does each invest?

Ans., A furnishes $\frac{s}{3+4m}$; B, $\frac{ms}{3+4m}$; C, $\frac{s(1+m)}{3+4m}$; and D, $\frac{s(1+2m)}{3+4m}$.

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21. At a certain election 943 men voted, and the candidate chosen had a majority of 65 votes. How many voted for each? *Ans.*, 439 and 504.

22. A farmer has two flocks of sheep, each containing the same number. From one he sells 39, and from the other 93, and then finds just twice as many in one flock as in the other. How many did each flock originally contain? *Ans.*, 147.

23. A person spends $\frac{1}{3}$ of his income in board and lodging, $\frac{1}{8}$ in clothing, and $\frac{1}{10}$ in charity, and has \$318 left. What is his income? *Ans.*, \$720.

24. From a bin of wheat $\frac{1}{2}$ was taken, and 20 bushels were added. After this $\frac{1}{3}$ of what was then in the bin was sold, and $\frac{1}{4}$ as much as then remained $+30$ bushels was put in, when it was found that the bin contained just $\frac{1}{2}$ as much as at first. How much did it contain at first?

Suggestion.—Calling x the amount in the bin at first, after taking out $\frac{1}{2}$, there remains $\frac{1}{2}x$. To this add 20 bushels and there is in the bin $\frac{1}{2}x+20$. After selling $\frac{1}{3}$ of this $\frac{2}{3}$ remains, or $\frac{2}{3}(\frac{1}{2}x+20)$. $\frac{1}{4}$ of this is $\frac{1}{6}(\frac{1}{2}x+20)$. $\therefore$ $\frac{2}{3}(\frac{1}{2}x+20)+\frac{1}{6}(\frac{1}{2}x+20)+30=\frac{1}{2}x$. Whence $x=560$, the answer.

25. A person has a hours at his disposal; how far may he ride in a coach which travels b miles per hour, and yet have time to return on foot walking at c miles per hour?

Ans., $\frac{abc}{b+c}$.

26. After paying $\frac{1}{n}$th of my money, and then $\frac{1}{m}$th of what remained, I had \$$a$ left. How much had I at first?

Equation, $x-\frac{1}{n}x-(x-\frac{1}{n}x)\frac{1}{m}=a$.

Ans., $\frac{amn}{mn-m-n+1}$.

27. A boy, being asked his age, replied, 11 years are 7 years more than $\frac{2}{5}$ of my age. How old was he?

Statement.—Let $x=$ his age. Then $\frac{2}{5}x+7=11$. $\therefore x=10$.

28. A boy, being asked how many sheep his father had, replied, that 40 were 5 less than $\frac{3}{4}$ of his father's flock. How many sheep had his father? *Ans.*, 60.

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29. If A can perform a piece of work in 10 days, and B in 8 days, in what time will they perform it together?

Statement.—Let $x =$ the number of days. Then since A can do the work in 10 days, in 1 day he will do $\frac{1}{10}$, and in x days $\frac{x}{10}$ of it. In like manner B will do $\frac{x}{8}$. Hence $\frac{x}{10} + \frac{x}{8} =$ (all the work) or 1. This 1 sometimes troubles pupils. But let them consider that if two of us do a piece of work, one doing $\frac{2}{3}$ and the other $\frac{1}{3}$, *if the work is all done*, the sum of the parts done is 1.

30. There is a certain piece of work which A and B can do in 8 days; but A and C can do it in 6 days, or B and C in 10 days. How long would it take any one of them to do it alone? How long if all work?

Suggestion. $\frac{1}{6} - \frac{1}{8}$ is the *difference* between what B and C can do in a day; and $\frac{1}{10}$ is the *sum* of what they can do in a day. (See **33.**)

Ans., A in $10\frac{10}{23}$ days, C in $14\frac{2}{17}$, B in $34\frac{2}{7}$, and all in $5\frac{5}{47}$.

31. A, B, and C can do a piece of work in 4 days, which A alone can do in 12, and B alone in 8 days. C begins the work alone, and is joined after 2 days by A, and they work together 2 days more. A and C being then called off, how long will it take B to finish the work? *Ans.*, $5\frac{1}{3}$ days.

32. A performs $\frac{2}{7}$ of a piece of work in 4 days; he then receives the assistance of B, and the two together finish it in 6 days. Required the time in which each could have done it alone?

Suggestion.—How much does A do in 1 day? Let $x =$ the time B would require to do it all. The equation is $\frac{6}{14} + \frac{6}{x} = \frac{5}{7}$. A could do it in 14, and B in 21 days.

33. A vessel can be emptied by 3 taps; by the first alone it could be emptied in 80 minutes, by the second in

200 minutes, and by the third in 5 hours. In what time will it be emptied if all the taps are opened at once?

Ans., 48 minutes.

34. A can do a piece of work in a days; B in b days, and C the same piece in c days. In how many days will they finish it, when all work together?

Ans., $\dfrac{abc}{ab+bc+ac}$.

35. Three men A, B, and C are employed on a certain piece of work. A and B can do it in a days; A and C in b days, and B and C in c days. How long would it take each to do it alone? How long would it take them to do the work together?

Ans., A in $\dfrac{2abc}{bc+ac-ab}$ days, B in $\dfrac{2abc}{bc+ab-ac}$, C in $\dfrac{2abc}{ab+ac-bc}$, and all together in $\dfrac{2abc}{ab+ac+bc}$.

Suggestion.—Notice the singular symmetry of these answers. Such symmetry is common, and sometimes becomes very useful in complicated processes.

***34.* Scholium.**—It is not always expedient to use x to represent the number sought. The solution is often simplified by letting x be taken for some number from which the one sought is readily found, or by letting $2x$, $3x$, or some multiple of x stand for the unknown quantity. The latter expedient is often used to avoid fractions.

36. There is a fish whose head is 9 inches long; the tail is as long as the head and half the body, and the body is as long as the head and tail together. What is the length of the fish?

Ans., 72 in.

Suggestion.—Let $x =$ the length of the body; then $9+\frac{1}{2}x =$ the length of the tail. The equation is $x = 18+\frac{1}{2}x$. If $x =$ the whole length the equation is $9+\frac{x}{2}+9+\frac{x}{4} = x$.

37. A general whose cavalry was $\frac{1}{3}$ of his infantry, after a defeat found that before the battle $\frac{1}{12}$ of his infantry less 120, and $\frac{1}{12}$ of his cavalry plus 120 had deserted. After the battle he found $\frac{1}{4}$ of his whole original army in garrison, $\frac{3}{8}$ on the field, and that the rest of those engaged were either taken prisoners or slain. Now 300 plus the number slain and captured was $\frac{1}{2}$ the infantry he had at first. Of how many did his army consist originally?

Suggestion.—Let $x =$ the number of cavalry; then $3x =$ the infantry, and $4x =$ the whole army.

Ans., Whole army 3600 men; viz., 900 cavalry, and 2700 infantry.

38. A shepherd, in time of war, was plundered by a party of soldiers who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep more; another party took $\frac{1}{3}$ of the remainder and $\frac{1}{3}$ of a sheep; and a third party took $\frac{1}{2}$ of the last remainder and $\frac{1}{2}$ of a sheep, when he had but 25 sheep left. How many had he at first.

Suggestion.—Letting $12x =$ the number in his flock at first, $9x-\frac{1}{4}$ is what remained after the first plundering, $6x-\frac{1}{2}$ after the second, and $3x-\frac{3}{4}$ after the third. The equation is $3x-\frac{3}{4} = 25$. $\therefore\ 12x = 103$, the number in his flock at first. The pupil should notice that he wants the value of $12x$ instead of x.

39. A cask, A, contains 12 gallons of wine mixed with 18 gallons of water; another, B, contains 9 gallons of wine mixed with 3 gallons of water. How many gallons must be drawn from each to make a mixture of 7 gallons of wine and 7 of water?

Suggestion. $\frac{2}{5}$ of the mixture in A is wine, and $\frac{3}{5}$ water; and in B $\frac{3}{4}$ is wine and $\frac{1}{4}$ water. Let $x =$ the number of gallons to be drawn from A; then $14-x$ represents the number to be drawn from B.

Of the first, $\frac{2}{5}x$ is wine, and $\frac{3}{5}x$ is water; of the second $\frac{3}{4}(14-x)$ is wine, and $\frac{1}{4}(14-x)$ is water. But in this new mixture the wine and water are equal. $\therefore\ \frac{2}{5}x+\frac{3}{4}(14-x)=\frac{3}{5}x+\frac{1}{4}(14-x)$.

40. In the composition of a quantity of gunpowder the nitre was 10 lbs. more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole, and the charcoal 2 lbs. less than $\frac{1}{7}$ of the nitre. What was the amount of the gunpowder? *Ans.*, 69 lbs.

Suggestion. x being the whole, the nitre is $\frac{2}{3}x+10$, the sulphur $\frac{1}{6}x-4\frac{1}{2}$, and the charcoal $\frac{1}{7}(\frac{2}{3}x+10)-2$.

41. Several detachments of artillery divided a certain number of cannon balls. The first took 72, and $\frac{1}{9}$ of the remainder; the next 144 and $\frac{1}{9}$ of the remainder; the third 216, and $\frac{1}{9}$ of the remainder; the fourth 288, and $\frac{1}{9}$ of those that were left; and so on; when it was found that the balls had been equally divided. What was the number of balls and detachments?

Ans., 4608 balls, and 8 detachments.

42. A gentleman bequeathed his property as follows: To his eldest child he left \$1800, and $\frac{1}{6}$ of the rest of his property; to the second, twice that sum and $\frac{1}{6}$ of what then remained; to the third, three times the same sum and $\frac{1}{6}$ of the remainder, and so on; and by this arrangement his property was divided equally among his children. How many children were there, and what was the fortune of each? *Ans.*, 5, and \$9000 the fortune of each.

43. A wholesale druggist has two grades of attar of roses; the one cost 9 dollars per ounce, the other 5. He wishes to mix both grades together in such quantities that he may have 50 ounces, and each ounce, without profit or loss, may be sold for 8 dollars. How much must he take of each grade to make up this mixture?

Ans., $37\frac{1}{2}$ ounces of the best, $12\frac{1}{2}$ of the other.

44. Let the price of the best grade in the preceding problem $= a$ dollars, the price of the poorest $= b$ dollars, the number of ounces in the mixture $= n$, and the price of the mixture $= c$. How many ounces of each kind must he use?

Ans., $\frac{(a-c)n}{a-b}$ ounces of the poorest, and $\frac{(c-b)n}{a-b}$ of the other.

45. A father, who has three children, bequeaths his property in the following manner: To the eldest son he leaves $1,000, together with the 4th part of what remains; to the second he leaves $2,000, together with the 4th part of what remains after the portion of the eldest and $2,000 have been subtracted from the estate; to the third he leaves $3,000, together with the 4th part of what remains after the portions of the two other sons and $3,000 have been subtracted. The property is found to be entirely disposed of by this arrangement. What was the amount of the property? *Ans.*, $9,000.

46. A father, who has three children, bequeaths his property in the following manner: To the eldest son he leaves a sum a, together with the nth part of what remains; to the second he leaves a sum $2a$, together with the nth part of what remains after the portion of the eldest and $2a$ have been subtracted from the estate; to the third he leaves a sum $3a$, together with the nth part of what remains after the portions of the two other sons and $3a$ have been subtracted. The property is found to be entirely disposed of by this arrangement. What was the amount of the property?

Ans., $\frac{(6n^2-4n+1)a}{(n-1)^2}$.

47. A general arranging his troops in the form of a solid square, finds he has 21 men over, but attempting to add 1 man to each side of the square, finds he wants 200 men to fill up the square; required the number of men on a side at first, and the whole number of troops.

Result, 110 and 12121.

Query.—Show that this problem is the same as the following? The difference between the squares of two consecutive numbers is 221. What are the numbers?

48. Gold is 19¼ times as heavy as water, and silver 10½ times. A mixed mass weighs 4160 ounces, and displaces 250 ounces of water. What proportions of gold and silver does it contain?

Ans., Gold 3377 ounces; silver, 783 ounces.

THE TRANSLATION OF EQUATIONS INTO PRACTICAL PROBLEMS.

***35.* Cor.**—*Since the statement of a problem is expressing the conditions of the problem in an equation, or in equations (**30**), it follows, conversely, that an equation may be considered as the enunciation in algebraic language of the conditions of a practical problem.*

EXAMPLES.

Ex. **1.** Form problems of which $\frac{x}{3} - \frac{x}{5} = 6$ is the statement.

Suggestion.—Any number of problems may be stated which will meet the requirements. Thus, "What number is that whose third part exceeds its fifth by 6?" Again; "A man being asked his age said, 'When I was ⅓ as old as I now am, I was 6 years older than when I was ⅕ as old as I am now;' what was his age?" Or, again; "A man being engaged to work a certain number of days, continued in the service $\frac{4}{5}$ of the whole time, but during this time was sick, and lost a number of days equal to $\frac{1}{5}$ of the whole time, when finally he had to break the engagement, having actually worked but 6 days. What was the whole period engaged for?"

2. Form problems of which $x+10x=66$ is the statement.

3. Give similar translations of each of the following equations: $\frac{x}{2}+\frac{x}{4}+2=x$; $\frac{x}{3}+\frac{x}{6}=1$; $2x-\frac{x}{2}-\frac{4x}{5}=7$; $x+\frac{1}{4}x-760+600=2000$; $x-\frac{4x+8}{6}=8$.

4. Translate the equations $3x+x=16000$; $x+\frac{x}{2}+\frac{x}{3}-10=100$.

5. Enunciate problems which will give rise to the following equations: $x-\frac{x}{5}-\frac{x}{7}=92$; $x+x+2x+4x=80$.

[**Note.**—The following examples give rise to equations found under **Art. 26.**]

6. What number is that which gives the same quotient when 28 is added to its square root, and the sum divided by its square root $+4$, as when 38 is added to the same root, and this sum divided by the square root of the number $+6$?

7. A man has a certain number of square rods of land lying in a square; if 12 rods be added, the whole being kept in the form of a square, his plat is increased by 2 rods on a side. How much land has he?

8. What number is that to which if 12 be added its square root is increased by 2?

9. If 4 times a certain number be increased by 1, the square root of this sum + twice the square root of the number itself, divided by the difference of the same quantities is 9. What is the number?

10. There is a certain number to which if its square root be added, and the square root of the remainder be taken, and again the square root be subtracted from the number, and the square root of this remainder be taken, the difference between the results is $1\frac{1}{2}$ times the square root of the quotient of the number divided by the number + its square root. What is the number?

11. What number is that from which if 32 be subtracted, the square root of the difference is equal to the square root of the number $-\frac{1}{2}$ the square root of 32?

12. What number is that from which if a be subtracted, the square root of the difference is equal to the square root of the number $-\frac{1}{2}$ the square root of a?

Ans., $\frac{25a}{16}$.

13. What number is that whose square root $+2a$, divided by its square root $+b$, equals its square root $+4a$, divided by its square root $+3b$?

Ans., $\left(\frac{ab}{a-b}\right)^2$.

Simple Equations

WITH TWO UNKNOWN QUANTITIES.

SECTION II.

DEFINITIONS.

36. The preceding problems have all been solved by a single equation containing only *one* unknown quantity. In some of them several quantities have been sought, it is true, but we have managed to represent these quantities by the use of a single unknown quantity, x. There are, however, many problems in which this is not practicable. In such problems there are *two* or *more* quantities sought, and the conditions are such as to give rise to two or more equations.

Illustration.—To make this latter statement clear, consider the following problem: A says to B, "If $\frac{1}{5}$ of my age were added to $\frac{2}{3}$ of yours, the sum would be $19\frac{1}{3}$ years." But, says B to A, "If $\frac{2}{5}$ of mine were subtracted from $\frac{7}{8}$ of yours, the remainder would be $18\frac{1}{4}$ years." Required their ages. Here are two distinct quantities sought; viz., A's age and B's age. Suppose we represent A's age by x, and B's by y. Now notice that there are also two sets of conditions. 1st, the statement which A makes to B; and, 2nd, the statement which B makes to A. According to the 1st, we have the equation $\frac{1}{5}x+\frac{2}{3}y=19\frac{1}{3}$; and according to the 2nd, $\frac{7}{8}x-\frac{2}{5}y=18\frac{1}{4}$.

37. **Independent Equations** are such as express *different* conditions, and neither can be reduced to the other.

38. **Simultaneous Equations** are those which express different conditions of the same problem, and consequently the letters representing the unknown quantities signify the same things in each. Each equation of a set of simultaneous equations is, therefore, satisfied by the same values of the unknown quantities.

Illustration.—Thus in the example above the two equations $\frac{1}{5}x + \frac{2}{3}y = 19\frac{1}{3}$ and $\frac{7}{8}x - \frac{2}{5}y = 18\frac{1}{4}$ are *independent* equations, since they express *different conditions*, and neither can be produced from the other. But, since these conditions are of the same problem, so that x in the first equation means the same as x in the second, and y in the first, the same as y in the second, they are *simultaneous* equations. It is evident that the true values of x and y must *satisfy*, or *verify*, *both* equations. If, however, we were to write one equation from one problem, and one from *another*, while they would be *independent*, they would *not* be *simultaneous;* x and y would not mean the same things in the first equation as in the second. In fact, the equations would be so independent, that they would have nothing to do with each other.

39. **Elimination** is the process of producing from a given set of simultaneous equations containing two or more unknown quantities, a new set of equations in which one, at least, of the unknown quantities shall not appear. The quantity thus disappearing is said to be eliminated. (The word literally means *putting out of doors.* We use it as meaning *causing to disappear.*)

40. There are **Three Methods of Elimination** in most common use: viz., by *Comparison*, by *Substitution*, and by *Addition or Subtraction.* There is also a very elegant method by Undetermined Multipliers, which is worthy of more attention than it generally receives, but which will be reserved for the advanced course.

Scholium.—Any one of these methods will solve all problems; but some problems are more readily worked by one method than by another, while it is often convenient to use several of the methods in the same problem, especially when there are more than two unknown quantities.

ELIMINATION BY COMPARISON.

41. **Prob. 1.—Having given two independent, simultaneous, simple equations between two unknown quantities, to deduce therefrom by comparison a new equation containing only one of the unknown quantities.**

Rule.—*I. Find expressions for the value of the same unknown quantity from each equation, in terms of the other unknown quantity and known quantities.*

II. Place these two values equal to each other, and the result will be the equation sought.

Demonstration.—The first operations being performed according to the rules for simple equations with one unknown quantity, need no further demonstration.

2nd. Having formed expressions for the value of the *same* unknown quantity in both equations, since the equations are simultaneous this unknown quantity means the same thing in the two equations, and hence the two expressions for its value are equal. Q. E. D.

Scholium.—The resulting equation can be solved by the rules already given.

EXAMPLES

OF INDEPENDENT SIMULTANEOUS EQUATIONS.

Ex. 1. Given $4x+y=34$ and $4y+x=16$; to find x and y and verify the values.

Model Solution.

Operation.

$$\begin{array}{lll} (1) & 4x+y=34 & \therefore\ x=\dfrac{34-y}{4} \\ (2) & 4y+x=16 & \therefore\ x=16-4y \\ & \dfrac{34-y}{4}=16-4y & \therefore\ y=2 \\ & 4x+2=34 & \therefore\ x=8 \end{array}$$

Explanation.—Transposing y and dividing by 4 I have from the 1st equation, $x=\dfrac{34-y}{4}$. Transposing $4y$ in the 2nd equation I have $x=16-4y$. Now, since these equations are assumed to be simultaneous, x means the same thing in both; and since things that are equal to the same thing are equal to each other, $\frac{34-y}{4} = 16 - 4y$. From this equation I find $y=2$. Finally, since y is found to be 2, putting 2 for y in the first equation, I have $4x+2=34$; whence $x=8$.

Verification.—Substituting for x, 8, and for y, 2, in the 1st equation I have $32+2=34$; and in the 2nd, $8+8=16$, we see that both equations are satisfied.

2. Given $5x+4y=58$, and $3x+7y=67$, to find x and y, eliminating by comparison. Verify the results.

Results, $\begin{cases} x=6, \\ y=7. \end{cases}$

3. Given $11x+3y=100$ and $4x-7y=4$, to find x and y, eliminating by comparison. Verify the results.

Results, $\begin{cases} x=8, \\ y=4. \end{cases}$

4. Same as above, given $8-\frac{x+3}{4}=7-\frac{3x-2y}{5}$ and $4x-\frac{8-y}{3}=24\frac{1}{2}-\frac{2x+1}{2}$.

Suggestion.—Observe the effect of the — sign before the compound quantities. The value of y from the 1st is $y=\frac{5+7x}{8}$, and from the 2nd, $y=\frac{160-30x}{2}$. Hence $\frac{5+7x}{8}=\frac{160-30x}{2}$, and $x=5$, and $y=5$.

5. Given
$$\begin{aligned} x \quad &+ ay = b \\ dx \quad &+ \ \ y = c \end{aligned}$$

6. Given
$$\begin{aligned} ax \quad &+ \frac{y}{b} = m \\ cx \quad &- \ \ y = n \end{aligned}$$

7. Given
$$\begin{aligned} ax \quad &- by = m \\ cx \quad &+ dy = n \end{aligned}$$

8. Given $ax+by=m$, and $cx+dy=n$, to find x and y, eliminating by comparison.

Suggestion.—After having found the value of x or y by comparison, find the value of the other in the same way.

ELIMINATION BY SUBSTITUTION.

42. Prob. 2.—**Having given two independent, simultaneous, simple equations, between two unknown quantities, to deduce therefrom by substitution a single equation with but one of the unknown quantities.**

Rule.—*I. Find from one of the equations the value of the unknown quantity to be eliminated, in terms of the other unknown quantity and known quantities.*

II. Substitute this value for the same unknown quantity in the other equation.

Demonstration.—The first process consists in the solution of a simple equation, and is demonstrated in **Art. 20.**

The second process is self-evident, since, the equations being simultaneous, the letters mean the same thing in both, and it does not destroy the equality of the members to replace any quantity by its equal. Q. E. D.

EXAMPLES.

Ex. 1. Given the independent, simultaneous equations $\frac{x+y}{2}-\frac{x-y}{3}=8$, and $\frac{x+y}{3}+\frac{x-y}{4}=11$, to find x and y, eliminating by substitution. Verify the results.

Model Solution.

Operation.

$$(1)\quad \frac{x+y}{2}-\frac{x-y}{3}=8$$

$$3x+3y-2x+2y=48$$

$$x+5y=48$$

$$x=48-5y$$

$$(2)\quad \frac{x+y}{3}+\frac{x-y}{4}=11$$

$$(3)\quad \frac{48-5y+y}{3}+\frac{48-5y-y}{4}=11$$

$$\frac{48-4y}{3}+\frac{24-3y}{2}=11$$

$$96-8y+72-9y=66$$

$$-17y=-102$$

$$\therefore\quad y=6$$

$$(4)\quad \frac{x+6}{2}-\frac{x-6}{3}=8$$

$$\frac{x}{2}-\frac{x}{3}=3$$

$$\therefore\quad x=18$$

Explanation.—Taking equation (1) I clear it of fractions and solve it for x, finding that $x=48-5y$. Now if I take the 2d equation and substitute this value for x, I shall have a simple equation with only the unknown quantity y in it. This substitution does not destroy the equality, since the equations are simultaneous (**38**). Making the substitution I have $\frac{48-5y+y}{3}+\frac{48-5y-y}{4}=11$. Reducing this equation by the method for simple equations with one unknown quantity, I find $y=6$.

Finally, resuming (1) I substitute this value for y, which evidently does not destroy the equation, and have $\frac{x+6}{2}-\frac{x-6}{3}=8$. Solving (4), which is now a simple equation with one unknown quantity, I find $x=18$. (Instead of taking (1) in its first form it would be better, because so much shorter, to take its reduced form, $x=48-5y$. $\therefore x=18$.)

[**Note**.—The student should keep a sharp lookout for opportunities to effect such reductions of terms as are made above in the equations following (3) and (4). In the latter the process consists in observing that $\frac{x+6}{2}$ is $\frac{x}{2}+3$, and $-\frac{x-6}{3}$ is $-\frac{x}{3}+2$, hence the first member becomes $\frac{x}{2}+3-\frac{x}{3}+2$, and transposing the 3 and 2, we have $\frac{x}{2}-\frac{x}{3}=3$, all of which can readily be effected mentally.]

2. Given $3x-2y=1$ and $3y-4x=1$, to solve as above.
Result, $x=5$ and $y=7$.

Suggestion.—From the second, $y=\frac{1+4x}{3}$, hence the first becomes $3x-\frac{2+8x}{3}=1$. $\therefore x=5$. Taking the reduced form of the second, $y = \frac{1+4x}{3}$, and putting 5 for x, $y = 7$. Verify both equations.

3. Given $\frac{3x-5y}{2} + 3 = \frac{2x+y}{5}$, and $8 - \frac{x-2y}{4} = \frac{x}{2} + \frac{y}{3}$.

Result, $x = 12$ and $y = 6$.

4. Given $\frac{x}{2} + \frac{y}{3} = 1$, and $\frac{x}{3} + \frac{y}{4} = 1$. Verify.

Result, $x = -6$ and $y = 12$.

5. Given $\frac{a}{x} + \frac{b}{y} = m$, and $\frac{c}{x} + \frac{d}{y} = n$.

Suggestion.—If we clear these equations of fractions they will become quite complex. But multiplying the first by c and the second by a, we have $\frac{ac}{x} + \frac{bc}{y} = cm$, and $\frac{ac}{x} + \frac{ad}{y} = an$. From the former $\frac{ac}{x} = cm - \frac{bc}{y}$, which substituted in the latter for $\frac{ac}{x}$ gives $cm - \frac{bc}{y} + \frac{ad}{y} = an$, or $\frac{ad-bc}{y} = an - cm$. $\therefore y = \frac{ad-bc}{an-cm}$. This value of y may now be substituted in the first. To get the value of x from such an equation as $\frac{1}{x} = \frac{dm-bn}{ad-bc}$, simply divide 1 by each member, *i. e.*, take their reciprocals.

6. Given $\frac{m}{x} + \frac{n}{y} = 1$, and $\frac{n}{x} + \frac{m}{y} = 1$.

Result, $x = m+n$, and $y = m+n$.

7. Given $\frac{x}{a} + \frac{y}{b} = 1$, and $\frac{x}{3a} + \frac{y}{6b} = \frac{2}{3}$.

Result, $x = 3a$, and $y = -2b$.

8. Given $\frac{1}{x} - \frac{2}{y} = 5$, and $\frac{3}{x} + \frac{4}{y} = 7$.

9. Given $\frac{x-y}{3} = \frac{x+y}{5}$, and $x + \frac{1}{2}y = 3$.

ELIMINATION BY ADDITION OR SUBTRACTION.

43. **Prob. 3.—Having given two independent, simultaneous, simple equations between two unknown quantities, to deduce therefrom by Addition or Subtraction a single equation with but one unknown quantity.**

Rule.—*I. Reduce the equations to the forms* $ax + by = m$, *and* $cx + dy = n$.

II. If the coefficients of the quantity to be eliminated are not alike in both equations, make them so by finding their L. C. M. and then multiplying the members of each equation by this L. C. M. exclusive of the factor which the term to be eliminated already contains.

III. If the signs of the terms containing the quantity to be eliminated are alike in both equations, subtract one equation from the other, member by member. If these signs are unlike, add the equations.

Demonstration.—The first operations are performed according to the rules already given for clearing of fractions, transposition, and uniting terms, and hence do not vitiate the equations. The object of this reduction is to make the two subsequent steps practicable.

The second step does not vitiate the equations, since in the case of either equation, both its members are multiplied by the same number.

The 3rd step eliminates the unknown quantity, since, as the terms containing the quantity to be eliminated have the same numerical value, if they have the *same* sign, by *subtracting* the equations one will destroy the other, and if they have different signs, by *adding* the equations they will destroy each other. The result is a true equation, since, If equals (the two members of one equation) are added to equals (the two members of the other equation), the sums are equal. Thus we have a new equation with but one unknown quantity. Q. E. D.

EXAMPLES.

Ex. 1. Given $5x+6y=28$, and $x+4y=14$, to eliminate by addition or subtraction and find the values of x and y. Verify the results.

Model Solution.

Operation.

$$\begin{array}{ll}
(1) & 5x + 6y = 28 \\
(2) & x + 4y = 14 \\
 & \\
(3) & 10x + 12y = 56 \\
(4) & \underline{3x + 12y = 42} \\
(5) & 7x \qquad\quad = 14, \therefore x = 2. \\
 & \\
(6) & 2 + 4y = 14 \\
 & \qquad 4y = 12, \therefore y = 3.
\end{array}$$

Explanation.—The equations being in the required form need no reduction.

To eliminate y I make its coefficients alike in both equations by multiplying the members of (1) by 2, and of (2) by 3, thus obtaining (3) and (4). This does not destroy the equations by (Ax. 2).

Then subtracting the members of (4) from the corresponding members of (3) I have (5), which is a true equation since the members of (3) have been increased equally (Ax. 2).

From (5) I have $x=2$.

Finally, substituting 2 for x in (2) I have (6), which is a true equation, since the value of the members of (2) is not altered by this substitution (Ax. 1).

From (6) I find by previous methods $y=3$.

Verification.—Substituting 2 for x, and 3 for y in both of the original equations, I have

$$\begin{array}{ll}
(1) & 10 + 18 = 28 \\
(2) & 2 + 12 = 14
\end{array}$$

Whence I see that both equations are satisfied for $x=2$, and $y=3$.

2. Given $77x-12y=289$, and $55x+27y=491$, to find x and y.

Model Solution.

Operation.

$$\begin{array}{lll}
(1) & 77x - \ \ 12y = 289 & (2)\ 55x + \ \ 27y = 491 \\
 & 693x - 108y = 2601 & \quad\ 220x + 108y = 1964 \\
 & \underline{220x + 108y = 1964} & \\
(3) & 913x \qquad\quad = 4565 &
\end{array}$$

$$(3)\quad 913x = 4565$$
$$x = 5$$
$$(4)\quad 275 + 27y = 491$$
$$27y = 216$$
$$y = 8$$

Explanation.—Since these equations are of the required form, I have only to make the numerical values of the terms to be eliminated alike. I will eliminate y, since its coefficients are smaller than those of x and the process will not involve as large numbers. The L. C. M. of 12 and 27 is 108, hence I multiply (1) by 9, obtaining $693x - 108y = 2601$, and (2) by 4, obtaining $220x + 108y = 1964$. This process does not vitiate the equations, since, Equals multiplied by the same number give equal products. I now observe that the signs of $108y$ in the two equations are different, and consequently that by adding the corresponding members of the equations these terms will destroy each other and give an equation in x only. Adding gives a true equation, since, Equals added to equals give equal sums. I therefore have $913x = 4565$, or $x = 5$. Substituting this result in (2), I have $275 + 27y = 491$, whence $y = 8$.

44. **Scholium.**—It is *usually* expedient, in examples involving two unknown quantities, to find the value of the second by substitution; but this is by no means always so. The pupil should perform the examples in several ways, if he can discern no choice of ways at first, and then compare the methods with reference to practicability.

3. Given $y + \frac{5x+2y}{6} - \frac{3y-12+8x}{5} = 4 - \frac{15+2x-4y}{3}$, and $\frac{7x+13-5y}{4} + x = 2y - \frac{3x+2y-16}{3}$, to find x and y.

Suggestion.—Cleared of fractions and reduced to their proper form these equations become

$$6y + x = 34$$
$$45x - 31y = 25$$

Whence $x = 4$, and $y = 5$.

4. Given $1+\frac{25+5y}{6}-\frac{7x-6}{3}=10-\frac{3x-10+7y}{12}$ and $\frac{96-8x}{9}=5x-\frac{14+y}{3}$, to find the values of x and y.

Results, $x=3$ and $y=7$.

5. Given $\frac{11x-5y}{22}=\frac{3x+y}{32}$ and $8x-5y=1$, to find the values of x and y. *Results*, $x=7$ and $y=11$.

45. **Scholium.**—In practice it often requires considerable discrimination to determine which of the methods of elimination to employ. But, as any one method will solve all cases, the pupil need not hesitate too long in attempting to select the best one. If he sees any reason why one method will be better than another in the given case, he will of course use it; but, if no such ground for choice is apparent, it will often be well to try more than one method, and see if one is any more expeditious than another.

EXAMPLES FOR GENERAL PRACTICE.

Ex. **1.** Given $3x-5y=13$ and $2x+7y=81$, to find x and y. *Results*, $x=16$ and $y=7$.

2. Given $\frac{x}{3}+\frac{y}{4}=9$ and $\frac{x}{4}+\frac{y}{5}=7$, to find x and y.

Results, $x=12$ and $y=20$.

3. Given $\frac{4x-3y-7}{5}=\frac{3x}{10}-\frac{2y}{15}-\frac{5}{6}$ and $\frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20}-1=\frac{y-x}{15}+\frac{x}{6}+\frac{1}{10}$, to find x and y.

Results, $x=3$ and $y=2$.

4. Given $\frac{2y}{18}-\frac{8x-2}{36}=1-\frac{4+y}{3}+\frac{x-y}{6}$ and $7x=12y$, to find x and y.

Suggestion.—The first equation reduces at once to $7x=7+11y$. In this case the pupil will see that the three methods of eliminating x are almost identical. *Comparing* the values of $7x$, we have $12y=7+11y$; or we may call it *substituting* the value of $7x$ found in the second equation; *i. e.*, $12y$, for $7x$ in the first. *Subtracting* the second from the first we should have $0 = 7-y$.

5. Given $\frac{15}{x}+\frac{21}{y}=10$ and $\frac{20}{x}-\frac{6}{y}=2$, to find x and y.

Suggestion.—The second becomes by dividing by 2, $\frac{10}{x}-\frac{3}{y}=1$, and by multiplying by 7, $\frac{70}{x}-\frac{21}{y}=7$. Now adding this to the first we have $\frac{85}{x}=17$, or $x=5$ and hence $y=3$.

6. Given $\frac{a}{x}+\frac{b}{y}=m$ and $\frac{c}{x}-\frac{d}{y}=n$, to find x and y.

Suggestion. $\frac{ad}{x}+\frac{bd}{y}=dm$ and $\frac{bc}{x}-\frac{bd}{y}=bn$, $\therefore \frac{ad+bc}{x}=dm+bn$ and $x=\frac{ad+bc}{dm+bn}$. Instead of substituting this value of x, it will be less work to eliminate x from the two given equations as we did y. Thus we have $\frac{ac}{x}+\frac{bc}{y}=cm$ and $\frac{ac}{x}-\frac{ad}{y}=an$, and subtracting, $\frac{bc+ad}{y}=cm-an$, $\therefore y=\frac{bc+ad}{cm-an}$

7. Given $\frac{1}{2}x+\frac{1}{3}y=14$ and $\frac{1}{3}x+\frac{1}{2}y=11$, to find x and y, and verify.

Suggestion.—Do not clear of fractions.

8. Given $x-\frac{3y-2+x}{11}=1+\frac{15x+\frac{4y}{3}}{33}$ and $\frac{3x+2y}{6}-\frac{y-5}{4}=\frac{11x+152}{12}-\frac{3y+1}{2}$, to find x and y, and verify.

9. Given $8x-\frac{16+60x}{3y-1}=\frac{16xy-107}{5+2y}$, and $2+6y+9x=\frac{27x^2-12y^2+38}{3x-2y+1}$, to find the values of x and y.

Suggestion.—Multiply the 1st equation by $5+2y$, and reduce before multiplying by $3y-1$. Clear the 2nd of fractions. Whence $289y-340x=187$, and $15x+2y=36$. $\therefore x=2$, $y=3$.

10. Given $3x+6y+1=\frac{6x^2+130-24y^2}{2x-4y+3}$, and $3x-\frac{151-16x}{4y-1}=\frac{9xy-110}{3y-4}$, to find the values of x and y.

Result, $x=9$, and $y=2$.

11. Given $16x+6y-1=\frac{128x^2-18y^2+217}{8x-3y+2}$, and $\frac{10x+10y-35}{2x+2y+3}=5-\frac{54}{3x+2y-1}$, to find the values of x and y.

Result, $x=6$, and $y=5$.

12. Given $(x+5)(y+7)=(x+1)(y-9)+112$, and $2x+10=3y+1$, to find the values of x and y.

Result, $x=3$, and $y=5$.

13. Given $bcx=cy-2b$, and $b^2y+\frac{a(c^3-b^3)}{bc}=\frac{2b^3}{c}+c^3x$, to find the values of x and y. *Result*, $x=\frac{a}{bc}$, $y=\frac{a+2b}{c}$.

Suggestion.—From the 1st equation $x=\frac{y}{b}-\frac{2}{c}$. Substituting this in the 2nd, $b^2y+\frac{a(c^3-b^3)}{bc}=\frac{2b^3}{c}+\frac{c^3y}{b}-2c^2$. Whence, transposing and uniting, $\frac{b^3-c^3}{b}y=\frac{2(b^3-c^3)}{c}+\frac{a(b^3-c^3)}{bc}$ or $\frac{b^3-c^3}{b}y=\frac{a+2b}{c}\times\frac{b^3-c^3}{b}$, and $y=\frac{a+2b}{c}$. Substituting this value of y in the 1st equation and reducing, we find $x=\frac{a}{bc}$. These equations can be solved by a variety of methods, but the pupil should constantly exercise his inventive genius to discover the most expeditious and elegant solutions.

14. Given $2x+0.4y=1.2$, and $3.4x-0.02y=0.01$, to find the values of x and y. *Result*, $x=.02$, $y=2.9$.

15. Given $\begin{cases} x+y=18.73 \\ 0.56x+13.421y=763.4 \end{cases}$

Result, $\left.\begin{cases} x=-39.81, \\ y=58.54, \end{cases}\right\}$ nearly.

APPLICATIONS.

Ex. 1. There are two numbers, such, that three times the greater added to one-third the less is 36; and if twice the greater be subtracted from 6 times the less, and the remainder divided by 8, the quotient will be 4. What are the numbers?

Model Solution.

Operation. Let $x =$ the greater number,
and $y =$ the less number.

Then (1) $3x + \frac{1}{3}y = 36$

(2) $\dfrac{6y-2x}{8} = 4$

(3) $6y + 54x = 648$

(4) $\underline{6y - 2x = 32}$

$56x = 616$

$x = 11$

(5) $6y - 22 = 32$

$y = 9$

Explanation.—As there are *two unknown quantities* involved in this example, *i. e.*, the two numbers sought, I let x represent the greater and y the less. There are also two sets of conditions stated in the problem: 1st, 3 times the greater added to $\frac{1}{3}$ the less is 36. This, according to the notation, is $3x+\frac{1}{3}y=36$, which is the first equation. The 2d set of conditions is, that twice the greater is to be subtracted from 6 times the less, which is $6y-2x$, and this difference divided by 8, *i. e.*, $\dfrac{6y-2x}{8}$. This quotient is equal to 4. Hence the second equation, $\dfrac{6y-2x}{8}=4$.

2. Find two numbers, such, that if the first be increased by a, it will be m times the second; and if the second be increased by b, it will be n times the first?

Result, $\frac{a+bm}{mn-1}$, and $\frac{b+an}{mn-1}$.

3. What two numbers are those, to $\frac{1}{3}$ of the sum of which if I add 13, the result will be 17; and if from $\frac{1}{2}$ their difference I subtract 1, the remainder will be 2? Verify.

Ans., 9 and 3.

[**Note.**—In verifying the results in such examples as these, no attention should be paid to the equations; but the results should be tested directly by the statement. Thus, in this example, $\frac{1}{3}$ of the sum of 9 and 3 is 4. Adding 13 the result is, as the example requires, 17. Again $\frac{1}{2}$ the difference of 9 and 3 is 3. Subtracting 1, the remainder is 2, as required.]

4. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Ans., $\frac{4}{5}$.

Queries and Suggestions.—How many sets of conditions in this problem? What are they? How many unknown (required) quantities? What are they? *There must always be as many of one as of the other.* The unknown (required) quantities here are the numerator and the denominator of the fraction. If these are called respectively x and y the fraction is $\frac{x}{y}$. Now, by the first set of conditions, $\frac{2x}{y+7} = \frac{2}{3}$, and, by the second set, $\frac{x+2}{2y} = \frac{3}{5}$.

5. What fraction is that which becomes $\frac{3}{4}$ when its numerator is increased by 6, and $\frac{1}{2}$ when its denominator is diminished by 2?

Ans., $\frac{9}{20}$.

6. If 1 be added to the numerator of a certain fraction, its value is $\frac{1}{3}$; but if 1 be added to its denominator, its value is $\frac{1}{4}$. What is the fraction? Verify. *Ans.*, $\frac{4}{15}$.

7. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left hand digit; and if from the number itself you subtract 18, the digits will have changed places. What is the number? Verify. *Ans.*, 53.

Suggestion.—The two numbers sought are the two digits. Hence let $y =$ the units digit, and $x =$ the tens digit. The number then is $10x+y$. Just as when 6 is the units digit of a number and 5 the tens, the number is $10 \times 5 + 6$. Of course the number would not be represented by xy, for this would indicate the *product* of the digits. (See PART I., **30,** SECOND LAW, Scholium 1st.) The first conditions give $2x - y = 7$, and the second $10x + y - 18 = 10y + x$, *i. e.*, the units becomes the tens figure and the tens becomes the units.

8. A certain number of two digits contains the sum of its digits four times and their product twice. What is the number? *Ans.*, 36.

9. There is a number consisting of two digits; the number is equal to 3 times the sum of its digits, but if the number be multiplied by 3, the product equals the square of the sum of its digits. What is the number? Verify.

10. A number consisting of 2 digits, when divided by 4 gives a certain quotient and a remainder of 3; when divided by 9, gives another quotient and a remainder of 8. Now, the digit on the left hand is equal to the quotient which was obtained when the number was divided by 9; and the other digit is equal to $\frac{1}{17}$ of the quotient obtained when the number was divided by 4. What is the number? Verify.

11. A farmer parting with his stock, sells to one person 9 horses and 7 cows for 300 dollars; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

Ans., $24 and $12.

12. A son asked his father how old he was. His father answered him thus: If you take away 5 from my years, and divide the remainder by 8, the quotient will be $\frac{1}{3}$ of your age; but if you add 2 to your age, and multiply the whole by 3, and then subtract 7 from the product, you will have the number of the years of my age. What was the age of the father and son?

Ans., 53 and 18.

13. A farmer purchased 100 acres of land for $2450; for a part of the land he paid $20 an acre, and for the other part $30 an acre. How many acres were there in each part? Verify.

Scholium.—Very many such problems can be solved equally well by means of one or of two unknown quantities.

14. At a certain election 946 men voted for two candidates, and the successful one had a majority of 558. How many votes were given for each candidate? Verify.

15. A jockey has two horses and two saddles. The saddles are worth 15 and 10 dollars, respectively. Now if the better saddle be put on the better horse, the value of the better horse and saddle will be worth $\frac{4}{3}$ of the other horse and saddle. But if the better saddle be put on the poorer horse, and the poorer saddle on the better horse, the value of the better horse and saddle will be worth once and $\frac{2}{13}$ the value of the other. Required the worth of each horse?

Result., 65 and 50 dollars.

16. A sum of money was divided equally among a certain number of persons; had there been four more persons, each would have received one dollar less, and had there been four fewer, each would have received two dollars more than he did: required the number of persons, and what each received? Verify.

Suggestion. $\frac{x}{y} = \frac{x}{y+4} + 1$, and $\frac{x}{y} = \frac{x}{y-4} - 2$. Hence $xy + 4x = xy + y^2 + 4y$, and $xy - 4x = xy - 2y^2 + 8y$, or $4x = y^2 + 4y$, and $-2x = -y^2 + 4y$. Adding, $2x = 8y$.

17. A farmer hired a laborer for ten days, and agreed to pay him $12 for every day he labored, and he was to forfeit $8 for every day he was absent. He received at the end of his time $40. How many days did he labor, and how many days was he absent? Verify.

18. A boatman can row down stream a distance of 20 miles, and back again, in 10 hours, the current being uniform all the time; and he finds that he can row 2 miles against the current in the same time that he rows 3 miles with it. Required the time in going and returning.

Result, 4 and 6 hours.

Suggestion.—If x and y are the times of rowing down and up, respectively, at what *rate* does he row down? At what rate up? Twice one of these rates equals 3 times the other.

19. A and B together could have completed a piece of work in 15 days, but after laboring together 6 days, A was left to finish it alone, which he did in 30 days. In how many days could each have performed the work alone?

Ans., 50, and $21\frac{3}{7}$ days.

Suggestion.—If x represent the number of days A would require to do it alone, and y the number B would require, how much would each do in a day? How much both? How much would they do in 6 days? How much would remain to be done by A alone? How much would A do in 30 days? In resolving these equations do not clear of fractions.

20. Two pipes, the water flowing in each uniformly, filled a cistern containing 330 gallons, the one running during 5 hours, and the other during 4; the same two pipes, the first running during two hours, and the second three, filled another cistern containing 195 gallons. The discharge of each pipe is required. Verify.

21. If I were to enlarge my field by making it 5 rods longer and 4 rods wider, its area would be increased 240 square rods; but if I were to make its length 4 rods less, and its width 5 rods less, its area would be diminished 210 square rods. Required the present length, width, and area. Verify.

22. A farmer sells a horses, and b cows for $\$m$; and at the same prices a_1 horses, and b_1 cows for $\$m_1$; what is the price of each? Apply the results to *Ex.* 11. See **(30)** Part 1.

Ans., Of a horse $\$\frac{b_1m-bm_1}{ab_1-a_1b}$; of a cow $\$\frac{a_1m-am_1}{a_1b-ab_1}$.

[**Note.**—Observe the symmetry of such results. Thus, in these numerators the a and b change places and in the denominators the subscripts change letters.]

23. A man bought s acres of land for $\$m$. For a part he paid $\$a$ per acre, and for the rest $\$a_1$ per acre. How many acres in each part? Deduce from the *general* answer obtained in this case the particular answers to *Ex.* 13.

Ans., $\frac{m-a_1s}{a-a_1}$ and $\frac{m-as}{a_1-a}$ acres.

24. A waterman rows a given distance a and back again in b hours, and finds that he can row c miles with the current for d miles against it: required the times of rowing down and up the stream, also the rate of the current and the rate of rowing? *Ans.*, Time down, $\frac{bd}{c+d}$; time up, $\frac{bc}{c+d}$; rate of current, $\frac{a(c^2-d^2)}{2bcd}$; rate of rowing, $\frac{a(c+d)^2}{2bcd}$.

Deduce from these answers those of Ex. 18.

Simple Equations

WITH MORE THAN TWO UNKNOWN QUANTITIES.

SECTION III.

46. **Prob.**—**Having given several simple, simultaneous, independent equations, involving as many unknown quantities as there are equations, to find the values of the unknown quantities.**

Rule.—*I. Combine the equations two and two by either of the methods of elimination, eliminating by each combination the same unknown quantity, thus producing a new set of equations, one less in number, and containing at least one less unknown quantity.*

II. Combine this new set two and two in like manner, eliminating another of the unknown quantities.

III. Repeat the process until a single equation is found with but one unknown quantity.

IV. Solve this equation and then substitute the value of this unknown quantity in one of the next preceding set of equations, and there will result an equation containing another single unknown quantity, the value of which can therefore be found.

V. Substitute the two values now found in one of the next preceding set, and find the value of the remaining unknown quantity in this equation. Continue this process till all the unknown quantities are determined.

Scholium I.—If any equation of any set does not contain the quantity you are seeking to eliminate from the following set, this equation can be written at once in that set and the remaining equations combined.

Scholium 2.—In eliminating any unknown quantity from a particular set of equations, any one of the equations may be combined with each of the others, and the new set thus formed. But some other order may be preferable as giving simpler results.

Scholium 3.—It is sometimes better to find the values of all the unknown quantities in the same way as the first is found, rather than by substitution.

Demonstration 1.—The combinations of the equations give true equations because they are all made upon the methods of elimination already demonstrated.

2. That the number of equations can always be reduced to one by this process, is evident, since, if we have n equations and combine any one of them with each of the others, there will be $n-1$ new equations. Combining one of these $n-1$ new equations with all the rest there will result $n-2$. Hence $n-1$ such combinations will produce a single equation; and as one unknown quantity, at least, has disappeared from each set there will be but one left. Q. E. D.

EXAMPLES.

Ex. 1. Given

$$\begin{array}{lr} (1_1) & 7x-2z+3u=17, \\ (2_1) & t+4y-2z=11, \\ (3_1) & 5y-3x-2u=8, \\ (4_1) & -3u+2t+4y=9, \\ (5_1) & 3z+8u=33, \end{array}$$

to find the values of x, y, z, t, and u.

Model Solution.

Operation.

$$\left.\begin{array}{lr} (1_2) & t+4y-2z=11 \\ (2_2) & 2t+4y-3u=9 \\ (3_2) & 3z+8u=33 \\ (4_2) & 35y-6z-5u=107 \end{array}\right\} \text{2nd set, from which } x \text{ is absent.}$$

$$\left.\begin{array}{lr} (1_3) & 3z+8u=33 \\ (2_3) & 35y-6z-5u=107 \\ (3_3) & 4y-4z+3u=13 \end{array}\right\} \text{3rd set, from which } x \text{ and } t \text{ are absent.}$$

$$\left.\begin{array}{lr} (1_4) & 3z+8u=33 \\ (2_4) & 116z-125u=-27 \end{array}\right\} \text{4th set, from which } x,\ t, \text{ and } y, \text{ are absent.}$$

$$\begin{array}{lll} (1_5) & 1303u = 3909 & \therefore\ u = 3 \\ \hline (1_4) & 3z+24 = 33 & \therefore\ z = 3 \\ (3_3) & 4y-12+9 = 13 & \therefore\ y = 4 \\ (1_2) & t+16-6 = 11 & \therefore\ t = 1 \\ (3_1) & 20-3x-6 = 8 & \therefore\ x = 2 \end{array}$$

Explanation.—I notice that I have 5 equations with 5 unknown quantities. From these I wish to produce a new set of 4 equations from which one at least of the unknown quantities shall be eliminated. I observe that x does not appear in (2_1), (4_1), and (5_1), hence I write these as three of the 2d set of equations. Then eliminating x between (1_1) and (3_1) I have (4_2), and thus obtain the 2nd set of 4 equations containing only 4 unknown quantities.

Again, as t is contained in a less number of this set of equations than any one of the other unknown quantities, I eliminate it next; *i. e.*, I produce a 3rd set which does not contain it. As (3_2) and (4_2) do not contain t, I transfer them at once to the 3rd set; and then eliminating t between (1_2) and (2_2) this set is complete, having 3 equations with 3 unknown quantities.

Now eliminating y from this set by combining (2_3) and (3_3), and transferring (1_3), I have the 4th set of two equations with only 2 unknown quantities. Combining these two so as to eliminate z I find $u = 3$.

Finally, substituting 3 for u in (1_4), I find $z = 3$.
Substituting 3 for u and 3 for z in (3_3), I find $y = 4$.
Substituting 4 for y and 3 for z in (1_2), I find $t = 1$.
Substituting the values of y and u in (3_1), I find $x = 2$.

2. Given $\left\{\begin{array}{l} x+y = 30 \\ x+z = 25 \\ y+z = 15 \end{array}\right\}$ *Values*, $\left\{\begin{array}{l} x = 20, \\ y = 10, \\ z = 5. \end{array}\right.$

3. Given $\left\{\begin{array}{l} 8x-4y = 24-z, \\ 6x+y = z+84, \\ x+80 = 3y+4z. \end{array}\right\}$ *Values*, $\left\{\begin{array}{l} x = 12, \\ y = 20, \\ z = 8. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 3u+x+2y-z = 22 \\ 4x-y+3z = 35 \\ 4u+3x-2y = 19 \\ 2u+4y+2z = 46 \end{array}\right\}$ *Values*, $\left\{\begin{array}{l} u = 4, \\ x = 5, \\ y = 6, \\ z = 7. \end{array}\right.$

5. Given $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=124, \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=\ \ 94, \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=\ \ 76, \end{array}\right\}$ *Results,* $\left\{\begin{array}{l} x=\ \ 48, \\ y=120, \\ z=240. \end{array}\right.$

Suggestion.—This may be solved in the ordinary way by clearing of fractions, etc., but the following is far more elegant:

Dividing the 1st by 3 and the 2nd by 2 and subtracting, we have

$$\left.\begin{array}{l} \frac{y}{72}+\frac{z}{60}=\frac{17}{3} \\ \frac{y}{30}+\frac{z}{24}=14 \end{array}\right\} \text{2nd set.}$$

Subtracting ½ the 1st from the 3rd.... $\frac{y}{30}+\frac{z}{24}=14$

Or,
$$\frac{y}{360}+\frac{z}{5\cdot 60}=\frac{17}{15}$$
$$\frac{y}{360}+\frac{z}{24\cdot 12}=\frac{7}{6}$$

Subtracting $\frac{z}{24\cdot\underset{2}{\cancel{12}}\cdot\underset{5}{\cancel{25}}}=\frac{1}{\underset{1}{\cancel{30}}}$ or $\frac{z}{240}=1$, and $z=240$.

6. Given $\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c, \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=\frac{2}{a+b-c}, \\ y=\frac{2}{a-b+c}, \\ z=\frac{2}{b+c-a}. \end{array}\right.$

Suggestion.—Do not clear of fractions. Having found the value of one unknown quantity, do not get the others by substitution, but return to the original equations and get each in the same manner.

7. Given $\left\{\begin{array}{l} \frac{2}{x}+\frac{3}{y}-\frac{4}{z}=\frac{1}{12}, \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}=\frac{19}{24}, \\ -\frac{4}{x}+\frac{5}{y}+\frac{6}{z}=\frac{1}{2}. \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=\ \ 6, \\ y=12, \\ z=\ \ 8. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x+a=y+z, \\ y+a=2x+2z, \\ z+a=3x+3y. \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=\dfrac{a}{11}, \\ y=\dfrac{5a}{11}, \\ z=\dfrac{7a}{11}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} 9x+6y+4z+3v+2u \\ \quad +1=0, \\ 16x+4v+1=0, \\ 25x-5y+z+5v-u \\ \quad +1=0, \\ x+2y+4z-v-2u \\ \quad +1=0, \\ 4x-2y+z-2v+u \\ \quad +1=0. \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=-\dfrac{169}{924}, \\ y=-\dfrac{220}{924}, \\ z=\dfrac{89}{924}, \\ v=\dfrac{445}{924}, \\ u=\dfrac{113}{924}. \end{array}\right.$

10. Given $\left\{\begin{array}{l} u+v+x+y=10, \\ u+v+x+z=11, \\ u+v+y+z=12, \\ u+x+y+z=13, \\ v+x+y+z=14. \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=3, \\ y=4, \\ z=5, \\ v=2, \\ u=1. \end{array}\right.$

11. Given $\left\{\begin{array}{l} \dfrac{3y-1}{4}=\dfrac{6z}{5}-\dfrac{x}{2}+1\frac{4}{5}, \\ \dfrac{5x}{4}+\dfrac{4z}{3}=y+\dfrac{5}{6}, \\ \dfrac{3x+1}{7}-\dfrac{z}{14}+\dfrac{1}{6}=\dfrac{2z}{21}+\dfrac{y}{3}. \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=2, \\ y=3, \\ z=1. \end{array}\right.$

12. Given $\left\{\begin{array}{l} 11x-10y=\dfrac{12y-11z}{3}, \\ \dfrac{x+z-2y}{3}=\dfrac{z-y-1}{2}, \\ 3x=y+z+7. \end{array}\right\}$ *Values,* $\left\{\begin{array}{l} x=10, \\ y=11, \\ z=12. \end{array}\right.$

13. Given $$\left\{\begin{array}{l} \frac{4x+3y+z}{10}-\frac{2y+2z-x+1}{15} \\ \quad =5+\frac{x-z-5}{5}, \\ \frac{9x+5y-2z}{12}-\frac{2x+y-3z}{4} \\ \quad =\frac{7y+z+3}{11}+\frac{1}{6}, \\ \frac{5y+3z}{4}-\frac{2x+3y-z}{12}+2z \\ \quad =y-1+\frac{3x+2y+7}{6}. \end{array}\right\}$$ *Values,* $$\left\{\begin{array}{l} x=9, \\ y=7, \\ z=3. \end{array}\right.$$

14. Given $$\left\{\begin{array}{l} x+\frac{1}{2}y=10-\frac{1}{3}z, \\ \frac{1}{2}(x+z)=9-y, \\ \frac{1}{4}(x-z)=2y-7. \end{array}\right\}$$ *Values,* $$\left\{\begin{array}{l} x=7, \\ y=4, \\ z=3. \end{array}\right.$$

APPLICATIONS.

Ex. **1.** The sum of three numbers is 9. The sum of the first, twice the second, and three times the third is 22. The sum of the first, four times the second, and nine times the third is 58. What are the numbers? *Ans.*, 1, 3, and 5.

Suggestion.—How many unknown quantities? How many sets of conditions? What are they? Express the first in an equation,—the second,—the third.

2. Five persons, A, B, C, D, and E played at cards; after A had won half of B's money, B one-third of C's, C one-fourth of D's, and D one-sixth of E's, they each had $7.50. How much had each to begin with?

Ans., A, $2.75; B, $9.50; C, $8.25; D, $8; and E, $9.

3. There are 4 men, A, B, C, and D, the value of whose estates is $14,000; twice A's, three times B's, half of C's, and one-fifth of D's, is \$16,000; A's, twice B's, twice C's, and two-fifths of D's, is \$18,000; and half of A's, with one-third of B's, one-fourth of C's, and one-fifth of D's, is \$4,000. Required the property of each.

Ans., A's, \$2,000; B's, \$3,000; C's, \$4,000; D's, \$5,000.

4. A number is represented by three figures; the sum of these is 11; the figure in the place of units is double that in the place of hundreds, and when 297 is added to this number, the sum obtained is represented by the figures of this number reversed. What is the number? *Ans.*, 326.

Suggestion.—Letting x represent the hundreds figure, y the tens, and z the units, the number is represented by $100x+10y+z$. The number with the digits reversed is $100z+10y+x$.

5. A man worked for a person ten days, having his wife with him 8 days, and his son 6 days, and he received \$10.30 as compensation for all three; at another time he wrought 12 days, his wife 10 days, and son 4 days, and he received \$13.20; at another time he wrought 15 days, his wife 10 days, and his son 12 days, at the same rates as before, and he received \$13.84. What were the daily wages of each?

Ans., The husband 75 cts.; wife, 50 cts. The son, 20 cts. *expense* per day.

Suggestion.—The value of the quantity representing the son's wages is found to be *negative*. Therefore the son produced the opposite effect from wages; *i. e.*, he was an *expense*.

6. Three masons, A, B, C, are to build a wall. A and B, jointly, can build the wall in 12 days; B and C can accomplish it in 20 days, and A and C in 15 days. How many days would each require to build the wall, and in what time will they finish it, if all three work together?

Ans., A requires 20 days; B, 30; and C, 60; and all three require 10 days.

7. Three laborers are employed on a certain work. A and B, jointly, can complete the work in a days; A and C require b days, B and C require c days. What time does each one, working alone, require to accomplish the work, on the condition that each one, under all circumstances, does the same quantity of work? And in what time would they finish it, if they all three worked together?

Ans., A requires $\frac{2abc}{bc+ac-ab}$ days, B $\frac{2abc}{bc+ab-ac}$ days, and C $\frac{2abc}{ab+ac-bc}$ days. Jointly, they require $\frac{2abc}{ab+ac+bc}$ days.

Deduce from these results those of the preceding example.

8. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days, in how many days can each alone perform the same work?

Ans., A in $14\frac{34}{49}$ days, B in $17\frac{23}{41}$ days, and C in $23\frac{7}{31}$ days.

9. A gentleman left a sum of money to be divided among his four sons, so that the share of the oldest was $\frac{1}{2}$ of the sum of the shares of the other three, the share of the second $\frac{1}{3}$ of the sum of the other three, and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was found that the share of the oldest exceeded that of the youngest by $14. What was the whole sum, and what was the share of each person?

Ans., Whole sum, $120; oldest son's share, $40; second son's, $30; third son's, $24; youngest son's, $26.

SYNOPSIS.

- SIMPLE EQUATIONS.
  - DEFINITIONS.
    - Algebra.
    - Equation. Members.
    - Independent Equations. Simultaneous.
    - Transposition. Elimination.
    - Statement. Solution.
  - KINDS OF EQUATIONS.
    - Simple.
    - Quadratic.
    - Higher.
      - With one unknown quantity.
      - With more than one unknown quantity.
  - AXIOMS.
  - TRANSFORMATIONS.
    - 1. Clearing of fractions. Rule. *Dem.* Ill.
    - 2. Transposition. Rule. *Dem.* Ill.
    - 3. Uniting terms.
    - 4. Dividing by coefficient of unknown quan.
  - Prob. 1. To solve equations. Rule. *Dem.*
    - Practical suggestions.
  - Prob. 2. To free an equation of radicals. Rule. *Dem.*
    - Practical suggestions.
  - ELIMINATION.
    - Number of methods. Reason for several.
    - Prob. 1. By comparison. Rule. *Dem.*
    - Prob. 2. By substitution. Rule. *Dem.*
    - Prob. 3. By addition and subtraction. Rule. *Dem.*
    - Prob. 4. With several unknown quantities. Rule. *Dem.*
      - Sch. 1, 2, 3.

Test Questions.—Upon what principle is an equation cleared of fractions? How is it done? Upon what principle is elimination by addition and subtraction performed? What comparison? Substitution? Give the seven Practical Suggestions upon solving Simple Equations. The six upon freeing of Radicals. Give the reason for changing the signs of the terms of a fraction having a polynomial numerator, preceded by a minus sign, when clearing of fractions. What is the general method of precedure in stating a problem? Does the statement involve a knowledge of anything but algebra? Illustrate. Upon what principle may all the signs of an equation be changed? (This may be explained in at least four different ways.) Having given the sum and difference of two quantities, how are the quantities found? Prove it.

RATIO, PROPORTION AND PROGRESSION

CHAPTER II.

SECTION I.

RATIO.

47. Ratio is the relative magnitude of one quantity as compared with another of the same kind, and is expressed by the quotient arising from dividing the first by the second. The first quantity named is called the *Antecedent*, and the second the *Consequent*. Taken together they are called the *Terms* of the ratio, or a *Couplet*.

48. The Sign of ratio is the colon, :, the common sign of division, ÷, or the fractional form of indicating division.

Illustration.—The ratio of 8 to 4 is expressed 8 : 4, 8÷4, or $\frac{8}{4}$, any one of which may be read "8 is to 4," or "ratio of 8 to 4." The *antecedent* is 8, and the *consequent* 4. The sign : is an exact equivalent for ÷, and by many writers, especially the Germans, the former is used exclusively. The sign : is, probably, a mere modification of ÷, made by dropping the horizontal line, as unnecessary. Possibly the sign ÷ finds its analogy in the fractional form for expressing division, by considering the upper dot as symbolizing a dividend, and the lower a divisor.

49. Cor.—*A ratio being merely a fraction, or an unexecuted problem in Division, of which the antecedent is the* *numerator, or dividend, and the consequent the denominator, or divisor, any changes made upon the terms of a ratio produce the same effect upon its value, as the like changes do upon the value of a fraction, when made upon its corresponding terms. The principal of these are,*

1st. *If both terms are multiplied, or both divided by the same number, the value of the ratio is* NOT CHANGED.

Illustration.

$$\begin{aligned} 16 : 8 &= 2 \\ 4\times16 : 4\times8 &= 2 \\ \tfrac{16}{2} : \tfrac{8}{2} &= 2 \end{aligned}$$

2nd. *A ratio is* MULTIPLIED *by multiplying the antecedent or by dividing the consequent.*

Illustration.

$$\begin{aligned} 32 : 8 &= 4 \\ 2\times32 : 8 &= 8 \\ 32 : \tfrac{8}{2} &= 8 \end{aligned}$$

3rd. *A ratio is* DIVIDED *by dividing the antecedent or by multiplying the consequent.*

Illustration.

$$\begin{aligned} 24 : 6 &= 4 \\ \tfrac{24}{2} : 6 &= 2 \\ 24 : 2\times6 &= 2 \end{aligned}$$

50. **A Direct Ratio** is the quotient of the antecedent divided by the consequent, as explained above. (47.)

An **Indirect** or **Reciprocal Ratio** is the quotient of the consequent divided by the antecedent, *i. e.*, the *reciprocal* of the direct ratio.

Thus, the direct ratio of 6 to 3 is 2, but the *inverse* ratio is $\frac{3}{6}$ or $\frac{1}{2}$. When the word ratio is used without qualification it means *direct* ratio. The *inverse* or *reciprocal*, it will be seen, is the ratio of the reciprocals. Thus the inverse ratio of 8 to 4 is the ratio of $\frac{1}{8}$ to $\frac{1}{4}$, or $\frac{1}{2}$.

51. *A ratio is always* WRITTEN *as a direct ratio.*

Thus, the inverse ratio of a to b is $b : a$, or $\frac{1}{a} : \frac{1}{b}$, the latter being expressed as the direct ratio of the reciprocals.

52. A ratio of **Greater Inequality** is a ratio which is greater than unity, as $4:3$. A ratio of **Less Inequality** is a ratio which is less than unity, as $3:4$.

53. A **Compound Ratio** is the product of the corresponding terms of several simple ratios.

Thus, the compound ratio $a:b$, $c:d$, $m:n$, is $acm:bdn$. This term corresponds to *compound fraction*. A compound ratio is the same in effect as a compound fraction.

54. A **Duplicate Ratio** is the ratio of the *squares*, a **triplicate**, of the *cubes*, a **subduplicate**, of the *square* roots, and a **subtriplicate**, of the *cube* roots of two numbers. Thus, $a^2:b^2$, $a^3:b^3$, $\sqrt{a}:\sqrt{b}$, and $\sqrt[3]{a}:\sqrt[3]{b}$.

EXAMPLES.

Ex. 1. What is the ratio of $3am^3$ to $6am^2$?

Model Solution.—Since the ratio of two quantities is the quotient of the antecedent divided by the consequent, the ratio $3am^3:6am^2$ is $\frac{3am^3}{6am^2}$, or $\frac{1}{2}m$.

2. What is the inverse ratio of $a-b$ to a^2-b^2?

Ans., $a+b$.

3. What is the ratio of $\frac{2}{3}$ to $\frac{1}{2}$? Of $\frac{5}{6}:\frac{2}{3}$? Of $\frac{(a-x)^2}{a^2+x^2}:a-x$? Of $\frac{4c^2m}{3b^3y}$ to $\frac{2cm}{3b^4y^2}$? Of $\frac{x^2-y^2}{x^3+y^3}:\frac{x-y}{x^2-xy+y^2}$.

Answers to three, 1, 1¼, and 1⅓.

4. What is the triplicate ratio of 6 to 2, *Ans.*, 27.

5. What is the subduplicate ratio of 64 to 16? *Ans.*, 2.

6. What is the compound ratio of 3 to 4, 8 to 9, 2 to 6, and 4 to 2? *Ans.*, $4:9$, or $\frac{4}{9}$.

7. Reduce $360:315$ to its lowest terms.

Suggestion.—This is the same as reducing a fraction to its lowest terms. The result is $8:7$.

8. Reduce $a^3+2a^2x : a^2$ to its lowest terms.

Result, $a+2x:1$.

9. Which is the greater, 16 : 15, or 17 to 14?

Suggestion.—To compare two fractions, reduce them to a common denominator. On the same principle these ratios become 224 : 210, and 255 : 210. Otherwise perform the division and compare the quotients. By this method we have 1.06 +, and 1.21 +.

10. Which is greater, $a^2-b^2 : a-b$, or $a^2+2ab+b^2 : a+b$?

11. Which is least of the ratios 20 : 17, 22 : 18, and 25 : 23?

12. Which is greater, $a+2 : \frac{1}{2}a+4$, or $a+4 : \frac{1}{2}a+5$?

Ans., $a+4 : \frac{1}{2}a+5 > a+2 : \frac{1}{2}a+4$.

13. What is the compound ratio of 15 : 12, 6 : 7, and 9 : 4? *Ans.,* 135 : 56.

14. Compound the ratios $a^2-x^2 : a^2$, $a+x : b$, and $b : a-x$. *Result,* The duplicate ratio of $a+x$ to a.

15. Show that the compound ratio of $x+y : a$, $x-y : b$, and $b : \dfrac{x^2-y^2}{a}$ is 1.

16. Is the compound ratio of $3a+2 : 6a+1$, and $2a+3 : a+2$, a ratio of greater or of less inequality, if a is $-\frac{3}{2}$? If a is 2? If a is -2? *Ans.,*—Zero, Greater, Infinity.

17. Compound the following: 7 : 5, the duplicate of 4 : 9, and the triplicate of 3 : 2. *Result,* 14 : 15.

Suggestion. $\frac{7}{5} \times \frac{\cancel{4}\cdot\overset{2}{\cancel{4}}}{\underset{3}{\cancel{9}}\cdot\cancel{9}} \times \frac{\cancel{3}\cdot\cancel{3}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 14 : 15.$

18. Compound the sub-duplicate of $x^2 : y^2$, and the triplicate of $\sqrt[3]{x} : \sqrt[3]{y}$. *Result,* $x^2 : y^2$.

19. Compound the inverse ratio of $\sqrt{x}+\sqrt{y}$ to $x-y$, and the direct ratio $x+2\sqrt{xy}+y : \sqrt{x}+\sqrt{y}$.

Result, $x-y$.

20. Which is the greater, the inverse subtriplicate ratio of 8 to 64, or the direct duplicate ratio of 2 to 3?

PROPORTION.

SECTION II.

55. **Proportion** is an equality of ratios, the terms of the ratios being expressed. The equality is indicated by the ordinary sign of equality, =, or by the double colon : : .

Thus, $8:4=6:3$, or $8:4::6:3$, or $8\div4=6\div3$, or $\frac{8}{4}=\frac{6}{3}$ all mean precisely the same thing. A proportion is usually read thus: "as 8 is to 4 so is 6 to 3."

Scholium.—The pupil should practice writing a proportion in the form $\frac{a}{b}=\frac{c}{d}$, still *reading* it "a is to b as c is to d." One form should be as familiar as the other. He must accustom himself to the thought that $a:b::c:d$ means $\frac{a}{b}=\frac{c}{d}$ and *nothing more*. It will be seen that the language "8 is to 4 as 6 is to 3," means simply that $\frac{8}{4}=\frac{6}{3}$, for it is an abbreviated form for saying that "the relation which 8 bears to 4 is the same as (is equal to) that which 6 bears to 2;" that is, 8 is as many times 4 as 6 is times 3, or $\frac{8}{4}=\frac{6}{3}$.

56. **The Extremes** (outside terms) of a proportion are the first and fourth terms. The **Means** (middle terms) are the second and third terms. Thus, in $a:b=c:d$, a and d are the *extremes*, and b and c are the *means*.

57. **A Mean Proportional** between two quantities is a quantity to which either of the two bears the same ratio that the mean does to the other.

Thus, if m is a mean proportional between a and b, a bears the same ratio to m that m does to b; *i. e.*, $a : m :: m : b$.

58. **A Third Proportional** to two quantities is such a quantity that the first is to the second as the second is to this third (proportional).

Thus, in the last proportion, b is a third proportional to a and m. So, also, a is a third proportional to b and m.

Scholium.—Notice carefully the language used in the last two definitions. We do not say "a mean proportional *to*," but "a mean proportional *between*," two others. So, again, we say "a third proportional *to* two others." Moreover, it is necessary that the two others be taken in the order named in the statement. Thus, if y is a third proportional to m and n, $m : n :: n : y$. But, if y is a third proportional to n and m, $n : m :: m : y$.

59. A proportion is taken by **Inversion** when the terms of each ratio are written in inverse order. Thus, if $a : b :: c : d$, by inversion we have $b : a :: d : c$. It is to be observed that in inversion the means are made extremes, and the extremes means.

60. A proportion is taken by **Alternation** when the means are made to change places, or the extremes. Thus, $a : b :: c : d$ becomes by alternation either $a : c :: b : d$, or $d : b :: c : a$. The appositeness of the term *alternation* (taking every other one) is seen from the fact that the new order is obtained by taking the terms alternately; that is, 1st and 3rd, 2d and 4th; or 4th and 2nd, 3rd and 1st.

61. A proportion is taken by **Composition** when the sum of the terms of each ratio is compared with either term of that ratio, the same order being observed in both ratios; or when the sum of the antecedents and the sum of the consequents are compared with either antecedent and its consequent. Thus, if $a : b :: c : d$, by composition we have $a+b : a :: c+d : c$, or $a+b : b :: c+d : d$, or $a+c : b+d :: a : b$, or $a+c : b+d :: c : d$.

62. If the *difference* instead of the *sum* be taken in the last definition, the proportion is taken by **Division.**

63. Four quantities are *Inversely* or *Reciprocally Proportional* when the 1st is to the 2nd as the 4th is to the 3rd, or as the reciprocal of the 3rd is to the reciprocal of the 4th. Thus, if a and b are to each other inversely, or reciprocally, as m and n, $a : b :: n : m$, or what is the same thing, $a : b :: \frac{1}{m} : \frac{1}{n}$.

64. **A Continued Proportion** is a succession of equal ratios, in which each consequent is the antecedent of the next ratio. Thus, if $a : b :: b : c :: c : d :: d : e$. we have a continued proportion.

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65. **Prop. 1.**—*In a proportion the product of the extremes equals the product of the means.*

Demonstration.—If $a : b :: c : d$ then $ad = bc$. For $a : b :: c : d$ is the same as $\frac{a}{b} = \frac{c}{d}$, which cleared of fractions becomes $ad = bc$. Q. E. D.

66. **Cor. 1.**—*The square of a mean proportional equals the product of its extremes, and hence a mean proportional itself equals the square root of the product of its extremes.* For, if $a : m :: m : d$, by the proposition $m^2 = ad$. Whence extracting the square root of each member, $m = \sqrt{ad}$.

67. **Cor. 2.**—*Either extreme of a proportion equals the product of the means divided by the other extreme; and, in like manner, either mean equals the product of the extremes* *divided by the other mean.* For, if $a:b::c:d$, $ad=bc$. $\therefore\ d=\frac{bc}{a}$, $a=\frac{bc}{d}$, $b=\frac{ad}{c}$, and $c=\frac{ad}{b}$.

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68. Prop. 2.—*If the product of two quantities equals the product of two others, the two former may be made the extremes, or the means of a proportion, and the two latter the other terms.*

Demonstration.—Suppose $my=nx$. Dividing each member by xy, we have $\frac{m}{x}=\frac{n}{y}$; *i. e.*, $m:x::n:y$. In like manner dividing by mn we have $\frac{y}{n}=\frac{x}{m}$, *i. e.*, $y:n::x:m$.

Let the pupil determine how each of the following forms may be deduced from the relation $my=nx$.

1. $m:x::n:y$..........Given above.
2. $m:n::x:y$..........By what do you divide?
3. $y:n::x:m$..........Given above.
4. $x:y::m:n$..........Dividing mx by each member we have $\frac{x}{y}=\frac{m}{n}$.
5. $y:x::n:m$..........By what do you divide?
6. $x:m::y:n$..........How obtained?
7. $n:m::y:x$.......... " "
8. $n:y::m:x$.......... " "

TRANSFORMATIONS OF A PROPORTION.

69. Prop. 3.—*Proposition 1, together with the two principles that such changes in the terms of a proportion may be made, as,*

1. *Do not change the values of the ratios,*
2. *Change both ratios alike,*

are sufficient to determine in all cases what transformations are possible without destroying the proportion.

That these two principles are correct is evident from the nature of a proportion, as an equality of ratios.

EXAMPLES.

MULTIPLES.

Ex. 1. If $a:b::x:y$, prove that $ma:mb::x:y$.

Solution.—This change does not alter the value of the first ratio, and hence the equality of ratios remains.

2. If $a:b::x:y$ is $ma:mb::nx:ny$? Why? Is the value of either ratio changed? Why?

3. If $a:b::x:y$ is $ma:b::mx:y$? Is the value of either ratio changed? How?

Ans., This change does not destroy the proportion, because it multiplies both ratios by the same quantity.

4. If $a:b::x:y$, is $a:mb::x:my$? or $\frac{a}{m}:b::\frac{x}{m}:y$? or $\frac{a}{m}:b::x:my$?

5. If the first term of a proportion is multiplied by any number, in what four ways may it be compensated so as not to destroy the proportion? Does multiplying the third term by the same number compensate? Why? Does dividing the 4th term? Why? Does dividing the second term? Why?

6. If the third term of a proportion is divided by any number, in what ways can the change be compensated so as not to destroy the proportion? Give the reason in each case.

CHANGES IN THE ORDER OF TERMS.

7. If $a:b::x:y$ is $a:x::b:y$? How are the ratios changed?

Solution. $a:b::x:y$ is the same as $\frac{a}{b}=\frac{x}{y}$ (55). Now multiplying each member of this equality by $\frac{b}{x}$, we have

$$\frac{a}{b}\times\frac{b}{x}=\frac{x}{y}\times\frac{b}{x}$$

$$\text{or }\frac{a}{x}=\frac{b}{y},\text{ which is}$$

$$a:x::b:y.$$

Thus we see that $a:b::x:y$ is transformed into $a:x::b:y$ by multiplying both ratios by $\frac{b}{x}$. This does not destroy their equality by (**69**, 2).

69a. *Hence we see that we can change the order of the means without destroying a proportion.*

8. If $m:x::n:y$, is $n:m::y:x$? How are the ratios changed?

Ans., Yes. The first, $\frac{m}{x}$, is multiplied by $\frac{xn}{m^2}$, and the second, $\frac{n}{y}$, is multiplied by $\frac{y^2}{xn}$, and $\frac{xn}{m^2}=\frac{y^2}{nx}$, since $nx=my$.

9. If four quantities are in proportion are they in proportion by inversion? How are the ratios changed?

Solution. $a:b::c:d$ is the same as $\frac{a}{b}=\frac{c}{d}$, by the definition of a proportion. Now dividing 1 by each member of this equality I have

$$\frac{b}{a}=\frac{d}{c},\text{ which is}$$

$$b:a::d:c.$$

The substance of this is that if two quantities are equal, their reciprocals are equal.

69b. *Hence we see that we can take a proportion by inversion without destroying it.*

10. If $3a^2b:2b^2::6mx:10m^2x^2$, is $2:a^2::5mx:b^2$?

Solution.—Taking the proportion, and cancelling like factors from both terms of the same ratio, which does not change its value, and like factors from both antecedents, which divides both ratios and hence does not destroy the proportion, and like factors from both consequents, which divides both ratios, and hence does not destroy the proportion, we have

$$\cancel{3}a^2\cancel{b} : \overset{b^2}{\cancel{2}\cancel{b}^{\cancel{3}}} :: \overset{2}{\cancel{6}}\cancel{m}\cancel{x} : \overset{5}{\cancel{10}}m^{\cancel{2}}x^{\cancel{2}},$$

$$\text{or}\quad a^2 : b^2 :: 2 : 5mx.$$

Whence changing the order of the ratios $2 : 5mx :: a^2 : b^2$, and finally changing the order of the means (**69**, 2), we have

$$2 : a^2 :: 5mx : b^2.$$

11. If $\frac{2}{3}ax^2 : \frac{1}{2}by :: a^3x : b^2y$, show that $b : \frac{1}{2} :: a^2 : \frac{2}{3}x$.

COMPOSITION AND DIVISION.

12. If $a : b :: m : n$, show that $a+b : a :: m+n : m$.

Solution. $a : b :: m : n$ is the same as $\frac{a}{b} = \frac{m}{n}$. Now add $\frac{b}{b}$ to the first member and $\frac{n}{n}$ to the second, and we have $\frac{a}{b} + \frac{b}{b} = \frac{m}{n} + \frac{n}{n}$, or $\frac{a+b}{b} = \frac{m+n}{n}$; that is $a+b : b :: m+n : n$.

13. If $a : b :: x : y$, show that $a-b : b :: x-y : y$.

Suggestion.—Subtract from the first ratio $\frac{b}{b}$, and from the second $\frac{y}{y}$.

14. If $m : n :: x : y$, show that $m+n : m-n :: x+y : x-y$.

Suggestion.—By the method of *Ex.* 12, I have $\frac{m+n}{n} = \frac{x+y}{y}$,

and by that of *Ex.* 13, $\frac{m-n}{n} = \frac{x-y}{y}$.

Dividing the former by the latter, I have $\frac{m+n}{m-n} = \frac{x+y}{x-y}$;

that is, $m+n : m-n :: x+y : x-y$.

69c. *Hence we see that a proportion may be taken by composition, or by division, or by both at once, without destroying it.*

15. If $\frac{1}{2}a - x : \frac{1}{2}a + x :: b - y : b + y$, show that $2x : y :: a : b$.

16. If $a : b :: x : y$, does it follow that $a-y : b-x :: a : x$?

Ans., No.

17. From $a : b :: x : y$, prove that $ma+nb : ma-nb :: mx+ny : mx-ny$. Also that if $a : b :: c : d$, and $m : n :: x : y$, $am : bn :: cx : dy$, and $\frac{a}{m} : \frac{b}{n} :: \frac{c}{x} : \frac{d}{y}$.

18. If $a : b :: x : y$ is $a^2 : b^2 :: x^2 : y^2$? Is $a^n : b^n :: x^n : y^n$? Is $\sqrt{a} : \sqrt{b} :: \sqrt{x} : \sqrt{y}$? Is $a^{\frac{1}{3}} : b^{\frac{1}{3}} :: x^{\frac{1}{3}} : y^{\frac{1}{3}}$? Is $a^m : b^m :: x^m : y^m$ whether m is integral or fractional, positive or negative? Why is it that the ratios remain equal in each case? How are they changed?

MISCELLANEOUS.

19. If $a : b :: c : d$, show that $ma+nb : pa+qb :: mc+nd : pc+qd$.

Suggestion.—The ratios to be compared when reduced to a C. D. are, $\frac{acmp+bcnp+admq+bdnq}{(ap+bq)(cp+dq)}$ and $\frac{acmp+bcmq+adnp+bdnq}{(ap+bq)(cp+dq)}$. Now from the given proportion we learn that $ad=bc$. Therefore, exchanging them in the two middle terms of the first ratio, the ratios become identical.

This may also be shown as follows: Multiplying antecedents by m and consequents by n, $ma : nb :: mc : nd$. By composition $ma+nb : ma :: mc+nd : mc$, or multiplying both ratios by m, $ma+nb : a :: mc+nd : c$. By changing the places of the means $ma+nb : mc+nd :: a : c$. In like manner it may be shown that $pa+qb : pc+qd :: a : c$. $\therefore$ $ma+nb : mc+nd :: pa+qb : pc+qd$, or $ma+nb : pa+qb :: mc+nd : pc+qd$. The student should give the reason why each step does not vitiate the proportion, according to (**69**).

20. If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$ prove that $a : b :: c : d$.

Suggestion.—Performing the operations and reducing, $2ad-2bc = -2ad+2bc$, or $ad=bc$. Whence $\frac{a}{b}=\frac{c}{d}$, or $a : b :: c : d$.

This may also be proved by writing according to Prop. 2, $a+b+c+d : a-b+c-d :: a+b-c-d : a-b-c+d$. Comparing the sum of each antecedent and its consequent with their difference, $2a+2c : 2b+2d :: 2a-2c : 2b-2d$. Whence $a+c : a-c :: b+d : b-d$. Repeating the same processes we have $a : b :: c : d$.

21. If $\frac{a^2-x^2}{4a} = b$, show that $a-x : 2a :: 2b : a+x$. Produce other forms of proportion from the given relation. How many can be produced?

22. If $r = s\sqrt{\frac{1}{2}}$, show that $r : s :: 1 : \sqrt{2}$.

23. If $(a+x)^2 : (a-x)^2 :: x+y : x-y$, show that $a : x :: \sqrt{2a-y} : \sqrt{y}$.

Solution. $a^2+2ax+x^2 : a^2-2ax+x^2 :: x+y : x-y$,

$2a^2+2x^2 : 4ax :: 2x : 2y$, $\quad a^2+x^2 : x^2 :: 2a : y$,

$a^2 : x^2 :: 2a-y : y$, $\therefore\ a : x :: \sqrt{2a-y} : \sqrt{y}$.

Let the student give the reasons.

24. If $a : b :: c : d :: e : f :: g : h :: i : k$, etc., show that $(a+c+e+g+i+$, etc.$) : (b+d+f+h+k+$, etc.$) :: a : b$, or $c : d$, or $e : f$, etc. That is, in a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents, as any antecedent is to its consequent.

Solution. $\frac{a}{b} = \frac{a}{b}$ or $ab = ba$, $\quad \frac{a}{b} = \frac{c}{d}$ or $ad = bc$,

$\frac{a}{b} = \frac{e}{f}$ or $af = be$, $\quad \frac{a}{b} = \frac{g}{h}$ or $ah = bg$, $\quad \frac{a}{b} = \frac{i}{k}$ or $ak = bi$, etc.

Adding, $a(b+d+f+h+k+$, etc.$) = b(a+c+e+g+i+$, etc.$)$; whence $(a+c+e+g+i+$, etc.$) : (b+d+f+h+k+$, etc.$) :: a : b$ or (since $a : b = c : d$, etc.), as $c : d :: e : f$, etc.

25. Four given numbers are represented by a, b, c, d; what quantity added to each will give sums which are proportional.

Ans., $\frac{bc-ad}{a-b-c+d}$.

26. If four numbers are proportionals, show that there is no number which, being added to each, will leave the resulting four numbers proportionals.

PROGRESSION

SECTION III.

70. A Progression is a series of terms which increase or decrease by a common difference, or by a common multiplier.* The former is called an *Arithmetical*, and the latter a *Geometrical Progression.*

A Progression is *Increasing* or *Decreasing* according as the terms increase or decrease in passing to the right. The terms *Ascending* and *Descending* are used in the same sense as increasing and decreasing, respectively.

In an increasing Arithmetical Progression the common difference is added to any one term to produce the next term to the right; and in a decreasing progression it is subtracted.

In an increasing Geometrical Progression the constant multiplier by which each succeeding term to the right is produced from the preceding is more than unity; and in a decreasing progression it is less than unity. This constant multiplier in a Geometrical Progression is called the *Ratio* of the series.†

71. The character, .., is used to separate the terms of an Arithmetical Progression, and the colon, :, for a like purpose in a Geometrical Progression.

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* This is the common use of the term. It is also used to include what is called a Harmonical Progression.

† This is an unfortunate use of the term Ratio, inasmuch as it is at variance with its use in proportion. To harmonize the use of the term in proportion, with this use, may have led some writers to define *ratio*, as used in proportion, as the quotient of the consequent divided by the antecedent. But the definition has neither logic nor the common usage of authors, English or Continental, to support it. The French use *rapport* in proportion, and *raison* in progression.

Illustrations.

$1 \,..\, 3 \,..\, 5 \,..\, 7$, etc., is an Increasing Arithmetical Progression with a common difference 2, or $+2$.

$15 \,..\, 10 \,..\, 5 \,..\, 0 \,..\, -5$, etc., is a Decreasing Arithmetical Progression with a common difference -5.

$a \,..\, a \pm d \,..\, a \pm 2d \,..\, a \pm 3d$, etc., is the general form of an Arithmetical Progression, d being the common difference.

$2 : 4 : 8 : 16$, etc., is an Increasing Geometrical Progression with ratio 2.

$12 : 4 : \frac{4}{3} : \frac{4}{9} : \frac{4}{27}$, etc., is a Decreasing Geometrical Progression with ratio $\frac{1}{3}$.

$a : ar : ar^2 : ar^3 : ar^4$, etc., is the general form of a Geometrical Progression, r being the ratio, and greater or less than unity, according as the series is increasing or decreasing.

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72. There are *Five Things* to be considered in any progression; viz., the first term, the last term, the common difference or the ratio, the number of terms, and the sum of the series, either three of which being given the other two can be found, as will appear from the subsequent discussion.

ARITHMETICAL PROGRESSION

73. Prop. 1.—*The formula for finding the* nth, *or last term of an Arithmetical Progression; or, more properly, the formula expressing the relation between the first term, the* nth *term, the common difference, and the number of terms of such a series is*

$$l = a + (n - 1)\,d,$$

in which a is the first term, d the common difference, n the number of terms, and l the nth or last term, d being positive or negative according as the series is increasing or decreasing.

Demonstration.—According to the notation, the series is

$$a\,.\,.\,a+d\,.\,.\,a+2d\,.\,.\,a+3d\,.\,.\,a+4d\,.\,.\,a+5d, \text{ etc.}$$

Hence we observe that as each succeeding term is produced by adding the common difference to the preceding, when we have reached the nth term, we shall have added the common difference to the first term $n-1$ times; that is, the nth term, or $l=a+(n-1)d$. Q. E. D.

Scholium.—As this formula is a simple equation in terms of a, l, n, and d, any one of them may be found in terms of the other three.

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74. Prop. 2.—*The formula for the sum of an Arithmetical Progression, i. e., expressing the relation between the sum of the series, the first term, last term, and number of terms is*

$$s=\left[\frac{a+l}{2}\right]n,$$

s representing the sum of the series, a the first term, l the last term, and n the number of terms.

Demonstration.—If l is the last term of the progression, the term before it is $l-d$, and the one before that $l-2d$, etc. Hence, as $a\,.\,.\,a+d\,.\,.\,a+2d\,.\,.\,a+3d$ - - - - l, represents the series, $l\,.\,.\,l-d\,.\,.\,l-2d\,.\,.\,l-3d$ - - - - a, represents the same series reversed. Now, the sum of the first series is

$$s=a+(a+d)+(a+2d)+\text{ - - - }(l-2d)+(l-d)+l;$$

and reversed, $s=l+(l-d)+(l-2d)+\text{ - - - }(a+2d)+(a+d)+a.$

Adding, $2s=(a+l)+(a+l)+(a+l)+\text{ - - - }(a+l)+(a+l)+(a+l).$

If the number of terms in the series is n, there will be n terms in this sum, each of which is $(a+l)$; hence $2s=(a+l)n$, or $s=\left[\frac{a+l}{2}\right]n$. Q. E. D.

Scholium.—This formula being a simple equation in terms of s, a, l, and n, any one of the four can be found in terms of the other three.

75. Cor. 1.—*Formulas.*

(1) $\quad l=a+(n-1)d,\quad$ *and*

(2) $\quad s=\left[\frac{a+l}{2}\right]n,\quad$ *being two equations between the five quantities,* a, l, n, d, *and* s, *any two of these five can be found in terms of the other three.*

[**Note.**—It is not considered worth while to make separate cases out of the different problems which arise in the progressions, or to cumber the memory with the multiplicity of *formulas* which can be deduced from the two fundamental ones, but rather that these should be fixed in memory, and their use clearly understood.]

EXAMPLES.

Ex. **1.** The first term of an A. P. is 2, and the common difference 3, what is the 11th term? What the sum of the series?

Solution.—In the first case there are under consideration the first term, $a = 2$, the common difference, $d = 3$, the number of terms, $n = 11$, and the last term, which is the thing required. The relation between these is given in $l = a + (n-1)d$; in which by substituting the given values there results $l = 2 + (11-1)3 = 32$. In the second case the formula $s = \left[\frac{a+l}{2}\right]n$ gives the relation, in which by substituting the given values there results $s = \left[\frac{2+32}{2}\right]11 = 187$.

2. The first term of an A. P. is 8, the last term 203, and the common difference 5, what is the number of terms? What the sum of the series? *Ans.*, $n = 40$, $s = 4220$.

3. The first term of an A. P. is 8, the last term 203, and the number of terms 40, what are the common difference and the sum?

4. The last term is 1, the sum 1717, and the number of terms 34, what are the first term and the common difference?

Suggestion.—The equations are $1 = a + (34-1)d$, and $1717 = \left(\frac{a+1}{2}\right)34$, from which to find a and d. $a = 100$, and $d = -3$.

5. What is the sum of the numbers 1, 2, 3, 4, etc., to 1000?

6. The first term of an arithmetical progression is 1, and the number of terms 23, what must be the *common difference* that the sum of all the terms may be 100? What the last term?

7. If the first term of an arithmetical progression is 100, and the number of terms 21, what must the common difference be that the sum of the series may be 1260? What the last term? *Ans.*, —4.

8. Two persons, A and B, start from the same place together, and travel in the same direction. A goes 40 miles per day; B goes 20 miles the first day, and increases his rate of travel $\frac{3}{4}$ of a mile per day. How far will they be apart at the end of 40 days, and which will be in advance?

Ans., A will be 215 miles in advance of B.

9. The first term of an arithmetical progression is —7, the common difference —7, and the number of terms 101, what is the sum of the series? *Ans.*, —36057.

76. Cor. 2.—*The formula for inserting a given number of arithmetical means between two given extremes is* $d = \frac{l-a}{m+1}$, *in which* m *represents the number of means. From this* d, *the common difference, being found, the terms can readily be written.*

Demonstration.—If a is the first term and l the last, and there are m terms between, or m means, there are in all $m+2$ terms. Hence substituting in the formula $l=a+(n-1)d$ for n, $m+2$, we have $l=a+(m+1)d$. From this $d=\frac{l-a}{m+1}$. Q. E. D.

10. Insert 8 arithmetical means between 3 and 21.

Series, 5..7..9..11..13..15..17..19.

11. Insert 3 arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

Series, $\frac{3}{8}$..$\frac{5}{12}$..$\frac{11}{24}$.

12. What is the nth term of the series 1..3..5..7.., etc.? *Ans.*, $2n-1$.

13. What is the sum of n terms of the series 1..3..5..7.., etc.? *Ans.*, n^2.

14. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall during the tth second? *Ans.*, $(2t-1)a$.

15. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall in t seconds? *Ans.*, at^2.

16. A debt can be discharged in a year by paying \$1 the first week, \$3 the second, \$5 the third, and so on; required the last payment and the amount of the debt.

Ans., Last payment, \$103; amount, \$2704.

17. A person saves \$270 the first year, \$210 the second, and so on. In how many years will a person who saves every year \$180 have saved as much as he? *Ans.*, 4.

18. A board, 2½ inches wide at the narrow end, and 10 feet long, increases in width 1½ inches for every foot in length; what is the width of the wide end? *Ans.*, 17½ in.

19. If 100 oranges are placed in a line, exactly 2 yards from each other, and the first 2 yards from a basket; what distance must a boy travel, starting from the basket, to gather them up singly, and return with each to the basket?

Ans., 11 mi. 3 fur. 32 rd. 4 yd.

[**Note.**—For other examples involving the principles of Arithmetical Progression, see Problems after Quadratics, and also the subject of Interest.]

GEOMETRICAL PROGRESSION

77. Prop. 1.—*The formula for finding the* n*th, or last term of a Geometrical Progression; or, more properly, the formula expressing the relation between the first term, the* n*th term, the ratio, and the number* *of terms of such a series is* $l = ar^{n-1}$, *in which* l *is the last, or* nth *term,* a *the first term,* r *the ratio, and* n *the number of terms.*

Demonstration.—Letting a represent the first term and r the ratio, the series is $a : ar : ar^2 : ar^3 : ar^4 :$ etc. Whence it appears that any term consists of the first term multiplied into the ratio raised to a power whose exponent is one less than the number of the term. Therefore the nth term, or $l = ar^{n-1}$. Q. E. D.

78. Prop. 2.—*The formula for the sum of a Geometrical Progression, or expressing the relation between the sum of the series, the first term, the ratio, and the number of terms is*

$$s = \frac{ar^n - a}{r - 1},$$

in which s represents the sum, a the first term, r the ratio, and n the number of terms.

Demonstration.—The sum of the series being found by adding all its terms, we have,

$$s = a + ar + ar^2 + ar^3 + \text{ - - } ar^{n-3} + ar^{n-2} + ar^{n-1}, \text{ and multiplying}$$

$$\text{by } r,\ \underline{rs = \quad ar + ar^2 + ar^3 + \text{ - - } ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n.}$$

Subtracting $rs - s = ar^n - a$, or

$$(r-1)s = ar^n - a, \text{ and } s = \frac{ar^n - a}{r-1}. \quad \text{Q. E. D.}$$

Suggestion.—The student will notice that multiplying the first series by r, and placing the terms of the product under the like terms of the series, simply moves each term, when multiplied, one place to the right, so that however many terms there may be in the series, each will have a similar one in the product except the first term, a; and each term in the product will have a similar one in the series, except the last one, ar^n.

79. Cor. 1.—*Formulas*

$$(1) \qquad l = ar^{n-1}, \quad \textit{and}$$

$$(2) \qquad s = \frac{ar^n - a}{r - 1} \quad \textit{being two equations}$$

between the five quantities, a, l, r, n, *and* s, *are sufficient to determine any* TWO *of them when the others are given.*

80. Cor. 2.—*Since* $l = ar^{n-1}$, $lr = ar^n$, *which substituted in* (2) *gives* $s = \frac{lr - a}{r - 1}$; *which formula is often convenient.*

EXAMPLES.

Ex. 1. The first term of a geometrical progression is 2, the ratio 3, and the number of terms 6. What are the last term and the sum of the series?

Ans., $l = 2 \cdot 3^5 = 486$, $s = \frac{ar^n - a}{r - 1} = 728$.

2. The last term of a geometrical progression is 62500, the ratio 5, and the number of terms 7. What are the first term and the sum of the series?

Ans., $a = 4$, and $s = 78124$.

3. By saving 1 cent the first week, 2 cents the second week, 4 cents the third week, and so on, doubling the amount every week, how much is saved the last week of the year? *Ans.*, $22,517,998,136,852.48.

Suggestion.—This problem requires us to raise 2 to the 51st power. This is readily effected thus: the 3rd power of 2 is 8. The 3rd power multiplied by the 3rd power gives the 6th power; hence $8 \times 8 = 64$ is the 6th power. In like manner $64 \times 64 = 4096$ is the 12th power, and $4096 \times 4096 = 16777216$ is the 24th power; and, finally, $16777216 \times 16777216 \times 8$ is the 51st power.

4. What is the sum of 10 terms of the series $8 : 4 : 2 : 1 : \frac{1}{2}$; etc.?

Suggestion. $s = \frac{a(r^n - 1)}{r - 1} = \frac{8\left(\frac{1}{2^{10}} - 1\right)}{-\frac{1}{2}} = 2^4 \times \frac{2^{10} - 1}{2^{10}} = \frac{1023}{2^6} = \frac{1023}{64} = 15\frac{63}{64}$. Also, $l = ar^{n-1} = \frac{8}{2^9} = \frac{1}{2^6} = \frac{1}{64}$. In such cases the ingenious student will avoid unnecessary multiplications by suppressing factors.

5. If 4 is the first term, 324 the last, and 5 the number of terms, what is the ratio?

81. Cor. 3.—*The formula for inserting* m *geometrical means between* a *and* l *is* $r = \sqrt[m+1]{\frac{l}{a}}$.

6. Insert 5 geometrical means between 3 and 192.

The ratio is 2.

7. Find 4 geometrical means between $\frac{1}{2}$ and $\frac{1}{3}$.

Result, The ratio is $\left(\frac{2}{3}\right)^{\frac{1}{5}}$, whence the series becomes

$$\frac{1}{2} : \frac{1}{2^{\frac{4}{5}} \cdot 3^{\frac{1}{5}}} : \frac{1}{2^{\frac{3}{5}} \cdot 3^{\frac{2}{5}}} : \frac{1}{2^{\frac{2}{5}} \cdot 3^{\frac{3}{5}}} : \frac{1}{2^{\frac{1}{5}} \cdot 3^{\frac{4}{5}}} : \frac{1}{3}.$$

Scholium.—This and many other problems in Geometrical Progression, are more readily solved by means of logarithms. Many also require a knowledge of quadratic equations, and even of the higher equations. Some farther illustrations will be given in their proper place, especially in treating the subject of Interest.

82. Cor. 4.—*The formula for the sum of an Infinite Decreasing Geometrical Progression is* $s = \frac{a}{1-r}$.

Demonstration.—Since in a decreasing progression the ratio is less than unity, the last term, ar^{n-1}, is also less than the first term, and numerator and denominator of the value of $\frac{lr-a}{r-1}$, both become negative. Hence it is well enough to write the formula for the sum of such a series $s = \frac{a-lr}{1-r}$, that is, change the signs of both terms of the fraction. Now, if the terms of a series are constantly decreasing, and the number of terms is infinite, we can fix no value, however small, which will not be greater than the last, or than some term which may be reached and passed. Hence we are compelled to call the last term of such a series 0, which makes the formula $s = \frac{a}{1-r}$. Q. E. D.

Scholium.— Decimal Repetends afford illustrations of such series. Thus .333+, is $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$, etc., to infinity. Again, .5434343+ is, $\frac{5}{10}$ + the series $\frac{43}{1000} + \frac{43}{100000} + \frac{43}{10000000} +$, etc., to infinity.

8. Find the sum of the series $1 : \frac{1}{2} : \frac{1}{4}$: etc., to infinity. *Sum*, 2.

9. Required the sum of the series $1, \frac{1}{3}, \frac{1}{9}$, - - - to infinity. *Sum*, $1\frac{1}{2}$.

10. Find the value of .1212 - - - to infinity. *Sum*, $\frac{4}{33}$.

11. Find the value of .2333, etc., to infinity. *Sum*, $\frac{7}{30}$.

12. Find the value of .3411111, etc., to infinity. *Sum*, $\frac{307}{900}$.

13. Find the value of .323232, etc., to infinity. *Sum*, $\frac{32}{99}$.

14. Find the value of .20414141, etc., to infinity. *Sum*, $\frac{2021}{9900}$.

15. Suppose a body to move eternally in this manner; viz., 20 miles the first minute, 19 miles the second minute, $18\frac{1}{20}$ the third, and so on in geometrical progression. What is the utmost distance it can reach? *Ans.*, 400 miles.

16. What is the distance passed through by a ball, before it comes to rest, which falls from the height of 50 feet, and at every fall rebounds half the distance? *Ans.*, 150.

17. In the preceding problem, suppose the body falls $16\frac{1}{12}$ feet the first second, 3 times as far the next second, and 5 times as far the third second, and so on, how long will it be before it comes to rest?

Ans., $\frac{10}{193}\sqrt{579}\,(4+3\sqrt{2}) = 10.27657+$ seconds.

Suggestion.—To fall 50 *ft.* takes $\sqrt{\frac{50}{16\frac{1}{12}}}$ *secs.* To rebound 25 takes $\sqrt{\frac{25}{16\frac{1}{12}}}$ *secs.* and to fall 25 the same time. To rebound $12\frac{1}{2}$ *ft.* takes $\sqrt{\frac{12\frac{1}{2}}{16\frac{1}{12}}}$ and the same to fall. Hence the series giving the time is

$$T=\sqrt{\frac{50}{16\frac{1}{12}}}$$

$$+\,2\left\{\sqrt{\frac{25}{16\frac{1}{12}}}+\frac{1}{\sqrt{2}}\sqrt{\frac{25}{16\frac{1}{12}}}+\frac{1}{2}\sqrt{\frac{25}{16\frac{1}{12}}}+\text{etc., to infinity}:\right\}$$

SYNOPSIS.

- **RATIO, PROPORTION, AND PROGRESSION.**
  - **RATIO.**
    - **Definitions.**
      - Ratio.—Terms of.—Antecedent.—Consequent.—Couplet.
      - Direct.—Inverse.—Greater Inequality, less.
      - Compound ratio.—Duplicate, sub-duplicate, etc.
    - Sign of.
    - *Cor.*—Changes in terms of.
  - **PROPORTION.**
    - **Definitions.**
      - Proportion.—Extremes.—Means.
      - Mean proportional.—Third proportional.
      - Inversion.—Alternation.—Composition.—Division.
      - Inverse or reciprocal proportion.—Continued.
    - **Fundamental Propositions.**
      - *Prop.* 1.—*Cor.* 1.—*Cor.* 2.
      - *Prop.* 2.
      - *Prop.* 3.—Principles on which transformations are made.
    - **Transformations.**
      - Equi-multiples.—Why proportion not destroyed.
      - Chg. in order of terms.— " " "
      - Composition, or division.— " " "
      - Involution, or evolution.— " " "
  - **PROGRESSIONS.**
    - **Definitions.**
      - Progression.—Arithmetical.—Geometrical.
      - Increasing, or ascending.—Decreasing, or descending.
      - Common difference, positive, negative.
      - Ratio, greater than 1, less than 1.
    - Sign of Arithmetical Progression.—Of Geometrical.
    - Five things.—Given, required.
    - **A. P.**
      - Two fundamental formulas.—Produce them.—To insert means.
    - **G. P.**
      - Two fundamental formulas.—Produce them.
        - To insert means.
        - Sum of infinite series.

Test Questions.—Give the various changes which can be made upon the terms of a ratio and tell how the ratio is affected, and *Why?* State the various transformations which can be made upon a proportion without destroying it, and *Give the reason in each case.* Produce the two fundamental formulas of Arithmetical Progression. Also of Geometrical.

APPLICATIONS.

[**Note.**—Teacher and pupil should bear in mind that the object of this section is to teach the properties of Ratio and Proportion; hence all the operations should be performed upon the proportion. The proportion should be kept in the form of a proportion, and not reduced to an equation.]

Ex. 1. Divide 60 into two parts which are to each other as 2 : 3.

Suggestion.—Letting x and $60-x$ be the parts, $x:60-x::2:3$. Hence $x:60::2:5$, or $x:24::2:2$; and $x=24$. The pupil should give the reason for each transformation. What is the first transformation? Composition. Why does it not destroy the proportion? What the second transformation? Why does it not destroy the proportion?

2. A boy being asked his age said: John, who is 18, is older than I; but, if you add to my age $\frac{1}{2}$ of it, and from this sum subtract $\frac{1}{4}$ of my age, the result will be to John's age as 10 : 9. How old was the boy? Verify.

Operation.

$$\begin{aligned} x+\tfrac{1}{2}x-\tfrac{1}{4}x:18&::10:9,\\ \tfrac{5}{4}x:18&::10:9,\\ x:18&::\ \ 8:9,\\ x:18&::16:18,\\ \therefore\ x&=16. \end{aligned}$$

Let the pupil tell, in each instance, just *what* the transformation is, and why, according to (**69**), the proportion is not destroyed.

Verification. $16+\frac{16}{2}-\frac{16}{4}=20$, which is to 18 as 10 is to 9, the ratio being $\frac{10}{9}$.

3. Two brothers being asked their ages, the younger replied, my age is to my brother's as 2 to 3; and if you add 18 to mine and 2 to his, the sums will be as 3 to 2. What were their ages?

Suggestion.—To solve with *one* unknown quantity we may represent the younger brother's age by $2x$ and the older's by $3x$.

Then $\quad 2x+18:3x+2::3:2;$

Whence $\quad 2x+18:27x+18::3:18,$

$$2x+18:25x::3:15::1:5.$$

$$2x+18:x::5:1,$$

$$2x+18 : 2x :: 5 : 2,$$
$$18 : 2x :: 3 : 2,$$
$$9 : x :: 9 : 6.$$
$$\therefore\ x = 6,\ 2x = 12 \text{ and } 3x = 18.$$

[**Note.**—True, it gives a somewhat shorter solution of this example to put the first proportion immediately into the equation $4x+36 = 9x+6$, whence $5x = 30$ and $x = 6$. But the object is to become familiar with the properties of a proportion.]

4. A man's age when he was married was to his wife's as 3 to 2; but after 4 years, his age was to hers as 7 to 5. What were their ages when they were married?

Suggestion.—This may be solved with *one* unknown quantity, like the preceding, and that is the more elegant way. We may also use two. Thus, $x : y :: 3 : 2$, and $x+4 : y+4 :: 7 : 5$. From the former $x : \frac{3}{2}y :: 3 : 3$ $\therefore$ $x = \frac{3}{2}y$. Substituting in the latter $\frac{3}{2}y+4 : y+4 :: 7 : 5$. Whence $3y+8 : 2y+8 :: 7 : 5$, $y : 2y+8 :: 2 : 5$, $2y : 2y+8 :: 4 : 5$, $2y : 8 :: 4 : 1$, and $y : 1 :: 16 : 1$. $\therefore$ $y = 16$, and $x = \frac{3}{2}y = 24$.

5. A man is now 25 years old and his brother is 15. How many years before their ages will be as 5 to 4? Verify.

6. A man has two flocks of sheep, each containing the same number; from one he sold 80 and from the other 20, when the numbers in the flocks were as 2 to 3. How many were there in each flock in the first place? Verify.

7. It is known to every one that a small body near the eye hides a large one farther off; and it is a principle in optics so nearly axiomatic that we will take it for granted, that, in order to have the smaller body just cover the larger their distances from the eye must be in proportion to their breadths, or lengths. From this some very pleasing calculations can be made. The pupil may make the following:

1st. The breadth of a man's thumb is about 1 inch, and he can readily hold it at 2 feet from his eye; how far off is the man who is 5 feet 8 inches high, when the breadth of the man's thumb at 2 feet from his eye just covers the man?

Ans., 136 feet.

2nd. Wishing to know approximately the height of the top of a steeple from the ground, I found that my hand, which is 4 inches wide near the thumb, when held 2 feet from my eye, just covered the height of the steeple at a distance of 240 paces of 3 feet each. What was the height of the top of the steeple from the ground? *Ans.*, 120 feet.

8. What number is that to which if 1, 5, and 13 be severally added, the second sum will be a mean proportional between the other two? Verify.

9. What number is that whose $\frac{1}{3}$ increased by 2 is to its $\frac{1}{5}$ diminished by 1, as 6 is to $2\frac{1}{2}$? *Ans.*, 30.

10. The number of acres a farmer planted with corn is to the number he planted with potatoes, as $\frac{4}{3}$ to 1; but if he had planted 6 acres less of corn, and $\frac{1}{2}$ as many potatoes $+15\frac{1}{2}$ acres, the ratio would have been as $\frac{2}{5}$ to $\frac{4}{3}$. How many acres of each did he plant?

[**Note.**—The five following examples are designed to be solved by using two or more unknown quantities.]

11. Find two numbers, the greater of which shall be to the less, as their sum to 42; and as their difference is to 6.

Suggestion. Let x = one and y the other.

Then (1) $x : y :: x+y : 42$,

(2) $x : y :: x-y : 6$,

By equality of ratios $x+y : 42 :: x-y : 6$,

and $x+y : x-y :: 7 : 1$,

$$2x : 2y :: 8 : 6,$$

$$x : 2y :: 4 : 6,$$

$$x : \tfrac{4}{3}y :: 4 : 4,$$

$$\therefore\ x = \tfrac{4}{3}y.$$

Substituting in (2) $\frac{4}{3}y : y :: \frac{4}{3}y - y : 6$,

Or, $\frac{4}{3} : 1 :: \frac{1}{3}y : 6$,

$$4 : 1 :: y : 6$$

$$1 : 1 :: y : 24$$

$$\therefore\ y = 24,$$

$$x = \tfrac{4}{3}y = 32.$$

12. Two numbers have such a relation to each other, that if 4 be added to each, they will be in proportion as 3 to 4; and if 4 be subtracted from each, they will be to each other as 1 to 4. What are the numbers? *Ans.*, 5 and 8.

Suggestion. $x+4 : y+4 :: 3 : 4$ and $x-4 : y-4 :: 1 : 4$. Whence $x+1 : 3 :: y : 4$, or $\frac{4x+4}{3} : 1 :: y : 1$. $\therefore\ y = \frac{4x+4}{3}$. Substituting, $x-4 : \frac{4x+4}{3} - 4 :: 1 : 4$, $x-4 : 2x-4 :: 1 : 6$, $x-4 : x :: 1 : 5$, $4 : x :: 4 : 5$. $\therefore\ x = 5$. Substituting, $6 : 3 :: y : 4$, or $3 : 3 :: y : 8$. $\therefore\ y = 8$.

13. Find two numbers in the ratio of 2½ to 2, such that when each is diminished by 5, they shall be in the ratio of 1⅓ to 1. *Numbers*, 25 and 20.

14. There are two numbers which are to each other as 16 to 9, and 24 is a mean proportional between them. What are the numbers? *Ans.*, 32 and 18.

[**Note.**—The following examples may be solved by converting the proportion into an equation, at whatever stage of the solution it is found expedient.]

15. Find two numbers in the ratio of 5 to 7, to which two other required numbers in the ratio of 3 to 5 being respectively added, the sums shall be in the ratio of 9 to 13; and the difference of those sums = 16.

Numbers, 30 and 42, and 6 and 10.

16. A farmer hires a farm for \$245 per annum; the arable land being valued at \$2 an acre, and the pasture at \$1.40; now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres are there of each?

Ans., 98 acres of arable, and 35 of pasture.

17. The quantity of water which flows from an orifice is proportioned to the area of the orifice, and the velocity of the water. Now there are two orifices in a reservoir, the areas being as 5 to 13, and the velocities as 8 to 7, and from one there issued in a certain time 561 cubic feet more than from the other. How much water did each orifice discharge in this time? *Ans.*, 440 and 1001 cubic feet.

18. At an election for two members of parliament, three men offer themselves as candidates, and all the electors give single votes. The number of voters for the two successful ones are in the ratio of 9 to 8; and could the first have had seven more without changing the numbers which the others had, his majority over the second would have been to the majority of the second over the third as 12 : 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each?

Ans., 369, 328, and 300, respectively.

19. A man, driving a flock of geese and turkeys to market, in order to distinguish his own from any he might meet on the road, pulled 5 feathers out of the tail of each turkey, and 2 out of the tail of each goose, and upon counting them, found that the number of turkeys' feathers lacked 15 of being twice those of the geese. Having bought 20 geese and sold 15 turkeys by the way, he found that the number of geese was to the number of turkeys as 8 to 3. What was the number of each at first?

Ans., 45 turkeys, and 60 geese.

PROBLEMS OF PURSUIT.

20. A fox starts up 120 feet ahead of a hound at exactly ½ past 2 o'clock P. M.; the hound gives chase and gains 5 feet every 2 minutes. At what time will he overtake the fox?

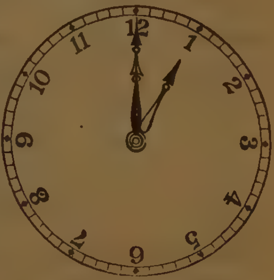
Statement.—Letting x be the time which will elapse before the hound overtakes the fox, the problem becomes: If a hound gain 5 feet in two minutes, how long will it take him to gain 120 feet? That is $5 : 120 :: 2 : x$. $\therefore\ x = 48$, and the hound overtakes the fox at 3 o'clock and 18 minutes.

21. A privateer espies a merchantman 10 miles to leeward at 11.45 A. M., and, there being a good breeze, bears down upon her at 11 miles per hour, while the merchantman can only make 8 miles per hour in her attempt to escape. After 2 hours chase the topsail of the privateer being carried away, she can only make 17 miles while the merchantman makes 15. At what time will the privateer overtake the merchantman? *Ans.*, At 5.30 P. M.

22. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to overtake the hare?

Suggestion. Let $3x =$ the number of the hound's leaps,
whence $4x =$ " " " hare's "
in the same time. Then $2 : 3 :: 3x : 4x + 50$. $\therefore\ x = 100$; and the hound takes 300 leaps.

23. The hour and minute hands of a clock are exactly together at 12 M. When are they next together?



Suggestion.—Measuring the distance around the dial by the hour spaces, the whole distance around is 12 spaces. Now, when the hour hand gets to 1, the minute hand has gone clear around, or over 12 spaces. But as the hour hand has gone *one* space, the minute hand has *gained* only 11 spaces. Now as the minute hand must *gain* an entire round, or 12 spaces, to overtake the hour hand, we have the question: If the minute hand gains 11 spaces in 1 hour, how long will it take to gain 12 spaces? $\therefore\ 11 : 12 :: 1$ hour : x hours; and $x = 1\frac{1}{11}$ hours, or 1 hour $5\frac{5}{11}$ minutes.

Scholium.—1. It is evident that the hands are together every $1\frac{1}{11}$ hours; hence to find at what time they are together between any two hours on the dial, we have only to multiply $1\frac{1}{11}$ by the number of the whole hours past 12 o'clock. Thus between 7 and 8 they pass each other at $7\frac{7}{11}$, or 7 o'clock and $38\frac{2}{11}$ minutes. Between 10 and 11 they pass each other at 10 o'clock $54\frac{6}{11}$ minutes.

24. At what time between 6 and 7 o'clock is the minute hand just $\frac{1}{4}$ of the circle in advance of the hour hand?

Suggestion.—The question is: If the minute hand gains 11 spaces in one hour, how long will it take it to gain $6\frac{1}{4}$ rounds, or 75 spaces? Or, if it gains 1 round in $\frac{1}{11}$ hours, how long will it take it to gain $6\frac{1}{4}$ rounds?

Ans., At $49\frac{1}{11}$ minutes past 6.

25. At what time between 4 and 5 is the hour hand of a watch just 20 minutes in advance of the minute hand?

Ans., At no time *between* these hours. The minute hand is within 20 minutes of the hour hand at 4 o'clock, and at $5\frac{5}{11}$ minutes past 5.

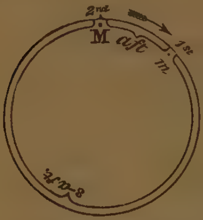
26. Before noon, a clock which is too fast, and points to afternoon time, is put back 5 hours and 40 minutes; and it is observed that the time before shown is to the true time as 29 to 105. Required the true time.

Suggestion.—Letting $x =$ the time pointed to, $x : x + 6\frac{1}{3} :: 29 : 105$. Observe that to turn the hands back 5h. 40m. is the same as to turn them forward 6h. 20m. $x =$ 2h. 25m., and the true time was 2h. 25m. + 6h. 20m., or 15m. before nine o'clock in the morning.

27. Two bodies move uniformly around the circumference of the same circle, which measures s feet. When they start, one is a feet before the other; but the first moves m and the second M feet in a second. When will these bodies pass each other the 1st time, when the 2nd, when the 3rd, etc., supposing that they do not disturb each other's motion, and go around the same way?

Suggestion.—1st. If $M > m$, the second gains $M - m$ feet a second, and having a feet to gain, overtakes the first, and does it in

$\frac{a}{M-m}$ seconds. The problem is then like the preceding, as the second gains a whole round every $\frac{s}{M-m}$ seconds. Hence the second passing is at $\frac{a+s}{M-m}$ from the starting, the third at $\frac{a+2s}{M-m}$, the fourth at $\frac{a+3s}{M-m}$, etc.



2nd. If $M < m$, the second is overtaken by the first after the first has gained $s-a$ feet, or in $\frac{s-a}{m-M}$ seconds; and in every $\frac{s}{m-M}$ seconds thereafter; that is, from the time of starting, in $\frac{2s-a}{m-M}$, $\frac{3s-a}{m-M}$, etc.

28. The earth makes a revolution around the sun in about 365 days and Mars in about 687 days. How long is it from the time at which these planets are together on the same side of the sun till they are next together? That is, how long does it take the earth to gain an entire revolution?

Ans., $687-365 : 365 :: 687 : x$, $\therefore$ $x = 780$ days, nearly.

[**Note.**—The time required by a planet to go around the sun is called its *Periodic Time*, or its year. When two planets are on the same side of the sun at the same time, they are said to be in *Conjunction.* The time from one conjunction to the next is the *Synodic Period.* The way in which this problem usually presents itself is this: We can *observe* when a planet is in conjunction with the earth, and thus knowing the time of the earth's revolution (a year), we can find the *Periodic Time* of the planet, or how long it takes to go around the sun.]

29. Calling the earth's *periodic time* $365\frac{1}{4}$ days, and observing Jupiter's *synodic period* to be about 399 days, how long is Jupiter in completing a revolution around the sun; that is, what is its *periodic time,* or length of its year?

Ans. $11\frac{11}{135}$ of our years.

30. Saturn's *synodic period* is about 378 days; what is its *periodic time?* *Ans.* $29\frac{11}{17}$ of our years.

PARTITIVE PROPORTION.

82a. **Partitive Proportion** is a term applied to the division of a number into parts which shall be in the ratio of given numbers.

31. Divide 350 into two parts which shall be to each other as 3 to 4. As 2 : 5. As 11 : 24.

Suggestion.—Let $3x$ represent one part, and $4x$ the other; whence $3x+4x=350$, $x=50$; and the parts are 150, and 200.

32. Divide a into two parts which shall be to each other as $m : n$.

The parts are $\frac{ma}{m+n}$, and $\frac{na}{m+n}$.

33. Divide 560 into three parts which shall be to each other as 17, 23, and 16. As 43, 2, and 11.

34. Divide a into three parts which shall be to each other as m, n, and r.

35. Divide 23 into two parts which shall be to each other as $\frac{2}{3}$ and $\frac{3}{5}$.

The parts are $12\frac{2}{19}$ and $10\frac{17}{19}$.

36. Divide a into two parts which shall be to each other in the ratio $\frac{1}{m}$ to $\frac{1}{n}$. As $\frac{r}{m} : \frac{s}{n}$.

The parts in the last case are $\frac{rna}{ms+rn}$, and $\frac{sma}{ms+rn}$.

BUSINESS RULES.

CHAPTER III.

SECTION I.

PERCENTAGE.

83. According to our definition, the Equation, of which it is the special province of Algebra to treat, is the grand instrument for investigating the relations of quantities. Now, in simple Percentage, there are four quantities to be compared; viz., the *Base*, the *Rate Per Cent.*, the *Percentage*, and the *Amount*, and the problem is, To discover and express in equations the relations between these four quantities so that if a sufficient number of them are given the others may be found.

[**Note.**—For Definitions see Practical Arithmetic, p. 225 *et seq.*]

84. **Prob. 1.—To express the relation between base, rate per cent., and percentage.**

Solution.—Let b represent the base, r the %, and p the percentage. Now $r\%$ means r 100ths of the base. Hence $r\%$ of b is $\frac{r}{100}$ times b, or $\frac{rb}{100}$. $\therefore p = \frac{rb}{100}$.

85. **Prob. 2.—To express the relation between rate per cent., amount or difference, and base.**

Solution.—Let s represent the sum or difference of the base and percentage. Then $s = b \pm \frac{rb}{100} = \frac{b(100 \pm r)}{100}$, the + sign to be used when the base is increased by the percentage, and the $-$ sign when the base is diminished by the percentage.

Scholium.—The two *formulæ*

$$(1)\ p=\frac{rb}{100} \text{ and}$$

$$(2)\ s=\frac{b(100\pm r)}{100}$$

expressing the relation between the four quantities b, r, p, and s, two of which must always be given to find the others, are in themselves sufficient for the solution of all problems in Simple Percentage.

EXAMPLES.

Ex. **1.** Bought a horse for \$840, and sold it for \$560. How much did I lose per cent.? *Ans.*, $33\frac{1}{3}\%$.

Suggestion.—Here b and s are given to find r. Hence formula (2) is to be used. And as there is loss involved, the $-$ sign is to be taken. Substituting in this formula, $560=\frac{840(100-r)}{100}$. Solving for r we have $\frac{56000}{840}=100-r$, or $\frac{200}{3}=100-r$; whence $r=100-\frac{200}{3}=\frac{100}{3}=33\frac{1}{3}$.

2. A number being increased by 2 equals 14. Required the increase per cent.? *Ans.*, $16\frac{2}{3}\%$.

Suggestion.—Formula (1) gives $2=\frac{rb}{100}$; and (2) gives $14=\frac{b(100+r)}{100}$. From which we are to find r. Multiplying (1) by 7, $14=\frac{7rb}{100}$. Whence $\frac{7rb}{100}=\frac{b(100+r)}{100}$ or $7r=100+r$, or $r=\frac{100}{6}=16\frac{2}{3}$.

3. A piece of cloth sold for \$779, cash, which was 5% off. Required the price of the cloth. *Price*, \$820.

4. Sold 40% of my wheat, and had remaining 981 bushels. How much had I at first? *Ans.*, 1635 bushels.

5. A man sold two horses at \$420 each; for one he received 25% more, and for the other 25% less than its value. Required his loss. *Loss*, \$56.

Suggestion.—Letting b and b_1 be the values of the horses, we have $420 = \frac{b(100-25)}{100}$, and $420 = \frac{b_1(100+25)}{100}$. $\therefore\ 75b = 125b_1$, or $b = \frac{5}{3}b_1$ and $b + b_1 = \frac{8}{3}b_1$, the value of the horses. Now b_1, found from the 2nd, is 336, and as $\frac{8}{3}b_1 - 840 =$ the loss, we have $896 - 840 = 56 =$ the loss.

Those who are not familiar with algebraic reasoning will prefer to find the values of b and b_1 from the two formulas, and add them together.

6. A man sold 72 turkeys, which was 32% of the number he had remaining. How many had he at first?
Ans., 297.

7. A farmer saved annually \$145½, which was 33⅓% of his annual income. Required his income? *Ans.*, \$436½.

8. A merchant having 400 barrels of cider, sold at one time 45% of it; at another time 20% of the remainder. How many barrels did he sell in all? *Ans.*, 224 *bbl.*

Operation.

$$p = \frac{rb}{100} = \frac{45 \times 400}{100} = 180.$$

$$p_1 = \frac{r_1 b_1}{100} = \frac{20 \times 220}{100} = 44.$$

$$\therefore\ p + p_1 = 224.$$

9. A housekeeper gave to her neighbor ⅛ of a pound of tea, and had ¾ of a pound remaining. What per cent. of her tea had she remaining? *Ans.*, 85⁵⁄₇%.

Operation. $\frac{3}{4} = \frac{r\frac{7}{8}}{100},\ 6 = \frac{7r}{100}.\ \therefore\ r = \frac{600}{7} = 85\frac{5}{7}.$

10. John has ½ of a dollar, and Henry has ⅗ of a dollar. What per cent. of John's money equals Henry's? What of Henry's equals John's? *Ans.*, 120%, 83⅓%.

Operation.

$$\frac{3}{5} = \frac{r\frac{1}{2}}{100},\ 6 = \frac{r}{20}.\quad \therefore\ r = 120.$$

$$\frac{1}{2} = \frac{r\frac{3}{5}}{100},\ 5 = \frac{3r}{50}.\quad \therefore\ r = \frac{250}{3} = 83\frac{1}{3}.$$

[**Note.**—For other examples in percentage, see Olney's PRACTICAL ARITHMETIC.]

SECTION II.

SIMPLE INTEREST AND COMMON DISCOUNT.

[**Note.**—For definitions see Practical Arithmetic.]

86. **Prob. 1.—To express the relation between principal, rate per cent., time, and interest.**

Solution.—Let p, r, t, and i represent respectively the principal, rate per cent., time and interest; t being in the denomination for which the rate per cent. is estimated. Thus, if the rate per cent. is rate per cent. per *year*, t is to be understood as *years;* if the rate per cent. is per month, t is months, etc.

Then as p is the base, the percentage for a unit of time is $\frac{rp}{100}$ (**84**); and for t units of time it is $\frac{trp}{100}$. $\therefore i = \frac{trp}{100}$.

87. **Prob. 2.—To express the relation between amount, principal, rate per cent., and time.**

Solution.—Since the amount is the sum of principal and interest, representing the amount by a, we have $a = p + i$. But $i = \frac{trp}{100}$; hence $a = p + \frac{trp}{100}$, or $a = p\,\frac{100+tr}{100}$. $\therefore p = \frac{100a}{100+tr}$.

88. **Scholium.**—This problem embraces the common rule for Discount (see Arithmetic, **383**). The pupil should be careful to understand the reasonableness of discount. For example, if I hold a note against Mr. B., which note is payable at any future time, *if the note is drawing interest for all money is worth, the Present Worth of the note at the time it was given was its face.* If the rate at which it is drawing interest is *less* than money is worth (or if it draws no interest) the Present Worth at its date is *less* than the face of the note. If it draws *no* interest, its Present Worth at *any time* is less than its face. Finally, if the rate which the note draws is *greater* than the market rate, the Present Worth of the note at its date is *more than its face.*

Scholium.—The two *formulæ* (1) $i = \frac{trp}{100}$ and

$$(2)\ a = p + i = p\,\frac{100+tr}{100},$$

are sufficient to solve all problems in Simple Interest and Common Discount.

EXAMPLES.

Ex. **1.** What is the interest on $250 for 1 *yr.* 10 *mo.* 15 *da.*, at 6% *per annum.* *Ans.*, $28.12½.

Solution.—In this example I am to consider principal, time, rate per cent. and interest, the latter of which is the unknown quantity. Formula (1) expresses the relation between these quantities. The rate per cent. being *per annum*, the time must be in years. 15 days $= \frac{15}{30} = .5$ of a month. 10.5 months $= \frac{10.5}{12} = .875$ of a year. ∴ 1 yr. 10 mo. 15 da. = 1.875 years. Now (1) gives $i = \frac{trp}{100}$

$$= \frac{1.875 \times \overset{3}{\cancel{6}} \times \overset{5}{\cancel{250}}}{\cancel{100}} = 28.12\tfrac{1}{2}.$$

2. What is the interest on $47.25 for 1 *yr.* and 6 *mo.*, at 6% *per annum?* *Ans.*, $4.2525.

Operation. $$i = \frac{\overset{.3}{\cancel{1.5}} \times \overset{3}{\cancel{6}} \times \overset{\overset{4.72\frac{1}{2}}{\cancel{9.45}}}{\cancel{47.25}}}{\underset{\underset{\underset{2}{\cancel{4}}}{\cancel{20}}}{\cancel{100}}} = 4.2525.$$

3. What is the interest on $145.50 for 1 *yr.* 9 *mo.* 24 *da.*, at 6% *per annum?* *Ans.*, $15.86 nearly.

Operation. $i = \frac{1.816 \times 6 \times 145.50}{100} = 15.86$ nearly.

4. What is the interest on $123.75 for 2 *yr.* 8 *mo.* 12 *da.*, at 6% *per annum?* *Ans.*, $20.0475.

5. What is the interest on $475 for 2 *yr.* 7 *mo.* and 20 *da.*, at 6% *per annum?* *Ans.*, $75.208⅓.

6. What is the interest on $340.60 for 4 *yr.* and 5 *mo.*, at 6% *per annum?* *Ans.*, $90.259.

7. What is the interest on $50.40 for 1 *yr.* and 10 *mo.*, at 7% *per annum?* *Ans.*, $6.468.

Suggestion.—In solving this example the operation upon paper consists simply in multiplying 12.833 by .504. The pupil should always reduce the written details of his arithmetical work to the minimum. Thus, in this case, he sees mentally that 10 mo. are .833 of a year. Hence $i = \frac{50.40 \times 1.833 \times 7}{100}$. But 1.833 × 7 he produces mentally, and writes 12.833. And cancelling the 100 from 50.40 makes it .504. Hence the only written work should be as in the margin. In practice, nothing but this multiplication should be written down.

$$\begin{array}{r} 12.833 \\ .504 \\ \hline 51332 \\ 64165 \\ \hline 6.467832 \end{array}$$

Solve the following by thus reducing the written work to a minimum. The answers are given as in practice in business.

8. What is the interest on $49.80 for 2 *yr.* and 11 *mo.*, at 7% *per annum?* *Ans.*, $10.17.

Operation. $i = \frac{49.80 \times 7 \times 2.9166}{100} = 2.9166 \times 7 \times .498 = 10.17.$

9. What is the interest on $95.40 for 3 *yr.* 9 *mo.*, at 8% *per annum?*

Operation. $i = \frac{95.40 \times 8 \times 3.75}{100} = 28\ 62.$ The operation should be performed mentally. Thus $8 \times 3.75 = 30$. Dropping the 0, and one 0 from the denominator, and for the 10 remaining in the denominator removing the decimal point in 95.40 so as to make it 9.54, we have simply to multiply 9.54 by 3. All of this should be done at a glance, without writing more than is given above.

10. What is the interest on $196 for 5 *yr.* 7 *mo.*, at 9% *per annum?* *Ans.*, $98.49.

11. What is the interest on $471.11 for 4 *yr.* 8 *mo.*, at 7½% *per annum?* *Ans.*, $164.89.

12. What is the interest on $18.60 for 3 *mo.* 12 *da.*, at 3% *per mo.?* *Ans.*, $1.90.

Operation. $i = \frac{3.4 \times 3 \times 18.60}{100} = 1.7 \times 3 \times .372 = .372 \times 5.1 = 1.90.$

13. What is the interest on $400 for $150 days, at 2½% *per month?* *Ans.*, $50.

Operation. $i = \dfrac{\overset{2}{\cancel{4}\cancel{0}\cancel{0}} \times 5 \times \dfrac{5}{\cancel{2}}}{\cancel{1}\cancel{0}\cancel{0}} = 50.$

14. What is the interest on $1,000 for 2 *mo.* 12 *da.*, at 1½% *per month?* *Ans.*, $36.

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15. What is the amount of $432.10 for 5 *yr.* 4 *mo.* 24 *da.*, at 7% *per annum?* *Ans.*, $595.43.

Operation. $a = p\dfrac{100 + tr}{100} = 432.10\,\dfrac{100 + 5.4 \times 7}{100} = 4.321 \times 137.8$ $= 595.4338.$

16. What is the amount of $325.25 for 2 *yr.* 9 *mo.* 12 *da.*, at 6½% *per annum?* *Ans.*, $384.09.

Operation. $a = 325.25\,\dfrac{100 + 2.78\frac{1}{3} \times 6.5}{100} = 384.09.$

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17. In what time will $13, at 6% *per annum*, give $0.975 interest? *Ans.*, 1 *yr.* 3 *mo.*

Solution.—Since principal, rate per cent., time and interest are compared $i = \dfrac{trp}{100}$ gives the relation. As time is required, I solve this equation for t, and have $t = \dfrac{100i}{rp}$. Substituting the given values,

$$t = \frac{\overset{50}{\cancel{1}\cancel{0}\cancel{0}} \times \overset{\overset{.025}{\cancel{.075}}}{\cancel{.975}}}{\cancel{6} \times \cancel{1}\cancel{3}} = 1\tfrac{1}{4}.$$

18. In what time will $45.25, at 6% *per annum*, give $1.81 interest? *Ans.*, 8 *mo.*

19. In what time will $70.50, at 9% *per annum*, give $31.725 interest? *Ans.*, 5 *yr.*

20. In what time will $140, at 7% *per annum*, give $10.861⅔ interest? *Ans.*, 1 *yr.* 1 *mo.* 9 *da.*

21. In what time will $48.50, at 6% *per annum*, amount to $56.187½? *Ans.*, 2 *yr.* 7 *mo.* 21 *da.*

Suggestion.—Subtract the principal from the amount to find the interest.

22. In what time will $248, at 6% *per annum*, amount to $282.224? *Ans.*, 2 *yr.* 3 *mo.* 18 *da.*

23. In what time will $700, at 9% *per annum*, amount to $712.35? *Ans.*, 2 *mo.* 10.5 + *da.*

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24. At what per cent. will $325 produce $3.25 interest in 2 months? *Ans.*, 6%.

Solution.—Same as above, finding r from the equation $i = \frac{trp}{100}$,

$$r = \frac{100i}{tp} = \frac{100 \times 3.25}{\frac{1}{6} \times 325} = 6.$$

25. At what per cent. will $40 produce $13.36 interest in 2 *yr.* 9 *mo.* 12 *da.*? *Ans.*, 12%.

26. At what per cent. will $125 produce $32.375 interest in 3 *yr.* 6 *mo.*? *Ans.*, 7⅖%.

27. At what per cent. will $124 produce 29.17\frac{4}{9}$ interest in 4 *yr.* 3 *mo.* 10 *da.*? *Ans.*, 5½%.

28. At what per cent. will $2,360.25 amount to $2,470.395 in 7 months? *Ans.*, 8%.

Suggestion.—Find the interest and then proceed as before.

29. At what per cent. will $230 amount to $249.83¾ in 11 *mo.* 15 *da.*? *Ans.*, 9%.

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30. What principal will in 3 *yr.* 8 *mo.* 15 *da.*, at 6% *per annum*, give $76.095 interest? *Ans.*, $342.

Solution.—Solving $i = \frac{trp}{100}$ for p, I have

$$p = \frac{100i}{tr} = \frac{100 \times 76.095}{6 \times 3.7\frac{1}{12}} = \frac{100 \times 76.095}{22.25} = 342.$$

31. What principal will in 4 *yr.* 9 *mo.* 18 *da.*, at 9% *per annum*, give $65.016 interest? *Ans.*, $150.50.

32. What principal will in 8 *yr.* 8 *mo.* 12 *da.*, at 5% *per annum*, give $147.9435 interest? *Ans.*, $340.10.

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33. How long will it take for $200, at simple interest, at 6% *per annum*, to amount to $500? *Ans.*, 25 years.

Solution.—I have under consideration p, r, a, and t, the latter of which is the unknown quantity. The relation between these is $a = p\frac{100+tr}{100}$. Solving this for t, I find

$$t = \frac{100(a-p)}{pr} = \frac{100(500-200)}{6 \times 200} = \frac{300}{12} = 25.$$

34. How long will it take $1, at simple interest, at 10% *per annum*, to amount to $100?

35. How long will it take $75, at interest at 5% *per annum*, to amount to $100?

89. **Cor.**—*To find the time required for a principal to double, triple, or become* n *times itself at any rate of simple interest, we have* $a = 2p, 3p,$ *or* np. *Hence the last formula becomes* $t = \frac{100(2p-p)}{pr} = \frac{100}{r}$ *for the time required to double. To triple we have* $t = \frac{100 \times 2}{r}$; *to quadruple,* $t = \frac{100 \times 3}{r}$; *to become* n *times itself* $t = \frac{100(n-1)}{r}$.

Scholium.—It appears that the time in such a case is independent of the principal, as might have been anticipated.

36. How long will it take $100 to become $500 at 8% *per annum?* *Ans.*, 50 years.

37. How long does it take a principal to double at 6% *per annum*; at 7%; at 5%?

38. Sold property amounting to $3,000, on a credit of 12 mo. without interest. Money being worth 8% *per annum*, what sum in hand is equivalent to the $3,000 under the contract? Verify it. *Ans.*, $2,777.77+.

Solution.—I am to find a sum which put at interest for 1 yr. at 8% will amount to $3,000. I therefore have given a, r, and t to find p. The relation of these is given in the formula $a = p\,\frac{100+tr}{100}$ from which $p = \frac{100a}{100+tr} =$ (in this case) $\frac{300000}{108} = 2777.77+.$

39. A debt of $500 will be due in 3 *yrs.* without interest. What is its present worth if money commands 6% *per annum*?

Ans., $423.73.

40. What *discount* should be allowed for the present payment of a note of $400, due 3 *yrs.* 5 *mo.* hence, the note not bearing interest, though money is worth 6% *per annum*?

Ans., $68.05.

41. A man buys a piece of property to-day for $5,000, giving his note with security at 12% *per annum*, payable 2 *yrs.* 9 *mo.* hence. What is the present worth of this note if money brings in market 6% *per annum*?

Ans., $5,708.13—.

Operation. $a = p\,\frac{100+tr}{100} = 50(100+12\times\frac{33}{12}) = 50\times 133 = 6650.$

$$p_1 = \frac{100a}{100+tr_1} = \frac{665000}{116.5} = 5708.15-.$$

42. A merchant buys $2,000 worth of goods on 90 days time, at 2% a month. He could have borrowed money at $1\frac{1}{2}$% a month. How many more goods could he have bought for the same money if he had borrowed?

Ans., $28.71 worth.

43. July 20th, 1869, I hold a note for \$500 dated April 6th, 1867, due Jan. 1st, 1872,, and bearing interest at 7% *per annum.* What is the present worth of this note, money being worth 10% *per annum?* *Ans.*, \$534.86+.

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90. Prob.—To find what each payment must be in order to discharge a given principal and interest in a given number of equal payments at equal intervals of time.

Solution.—Let p represent the principal, r the rate per cent., t one of the equal intervals of time, n the number of payments, (*i. e.*, nt is the whole time), and x one of the payments.

There will be as many solutions as there are different methods of computing interest on notes upon which partial payments have been made.

1st. *By the United States Court Rule.*—As the payments must exceed the interest in order to discharge the principal, this rule requires that we find the amount of p, for time t, at r per cent. This is done by multiplying by $1+\frac{rt}{100}$, and gives $p\left(1+\frac{rt}{100}\right)$. From this subtracting the payment x, the new principal is $p\left(1+\frac{rt}{100}\right)-x$. Again finding the amount of this for another period of time, t, and subtracting the second payment

$$p\left(1+\frac{rt}{100}\right)^2-x\left(1+\frac{rt}{100}\right)-x.$$

In like manner, after the third payment there remains

$$p\left(1+\frac{rt}{100}\right)^3-x\left(1+\frac{rt}{100}\right)^2-x\left(1+\frac{rt}{100}\right)-x.$$

After the 4th payment, the remainder is

$$p\left(1+\frac{rt}{100}\right)^4-x\left(1+\frac{rt}{100}\right)^3-x\left(1+\frac{rt}{100}\right)^2-x\left(1+\frac{rt}{100}\right)-x.$$

Finally, after the nth payment, we have

$$p\left(1+\frac{rt}{100}\right)^n-x\left(1+\frac{rt}{100}\right)^{n-1}-x\left(1+\frac{rt}{100}\right)^{n-2}-\text{-----------------}$$
$$x\left(1+\frac{rt}{100}\right)^2-x\left(1+\frac{rt}{100}\right)-x=0.$$

Whence

$$x=\frac{p\left(1+\frac{rt}{100}\right)^n}{1+\left(1+\frac{rt}{100}\right)+\left(1+\frac{rt}{100}\right)^2+\left(1+\frac{rt}{100}\right)^3 - - - \left(1+\frac{rt}{100}\right)^{n-1}}$$

This denominator being the sum of a geometrical progression whose first term is 1, ratio $\left(1+\frac{rt}{100}\right)$, and number of terms n, its sum is $\dfrac{\left(1+\frac{rt}{100}\right)^n-1}{\frac{rt}{100}}$. Hence $x=\dfrac{\frac{prt}{100}\left(1+\frac{rt}{100}\right)^n}{\left(1+\frac{rt}{100}\right)^n-1}$.

2nd. *By the Vermont Rule.*—The amount of the principal for the whole time is $p\left(1+\frac{rtn}{100}\right)$.

The amount of the 1st payment is - - - - - - - - - $x\left[1+\frac{rt}{100}(n-1)\right]$,

" " " 2nd " - - - - - - - - - $x\left[1+\frac{rt}{100}(n-2)\right]$,

" " " 3rd " - - - - - - - - - $x\left[1+\frac{rt}{100}(n-3)\right]$,

Etc., etc., - - - - - - - - - etc.

" " - - - - - - - - - "

The nth payment (with no interest) is - - - - - - x.

The sum of the amounts of these payments is

$$nx+\frac{rt}{100}x[(n-1)+(n-2)+(n-3) - - - - 1].$$

The series in the brackets being an arithmetical progression whose first term is $(n-1)$, common difference -1, last term 1, and number of terms $(n-1)$, its sum is $\left(\frac{n-1}{2}\right)n$. Hence the sum of the payments is $nx+\frac{rt}{100}x\left(\frac{n-1}{2}\right)n$, or $x\left[n+\dfrac{\frac{nrt}{100}(n-1)}{2}\right]$. But by the condition this sum equals the amount of the principal; consequently

$$x\left[n+\frac{\frac{nrt}{100}(n-1)}{2}\right]=p\left(1+\frac{nrt}{100}\right),\text{ and}$$

$$x=\frac{2p\left(1+\frac{nrt}{100}\right)}{2n+\frac{nrt}{100}(n-1)}.$$

Scholium.—If the payments are made annually, $t = 1$. And letting $r' = \frac{r}{100}$, *i. e.*, letting the rate per cent. be expressed decimally, the formulas become

$$\textit{By the U. S. Rule} \quad x = \frac{pr'(1+r')^n}{(1+r')^n - 1};$$

$$\textit{By the Vermont Rule} \quad x = \frac{2p(1+r'n)}{2n + r'n(n-1)}.$$

44. What must be the annual payment in order to discharge a note of $5,000, bearing interest at 10% per annum, in 5 equal payments?

Ans., By the U. S. Rule, $1,318.99 within a half cent. By the Vermont Rule, $1,250.

Query.—What occasions the great disparity between the payments required by the different rules?

45. What annual payment is required to discharge a note of $300, bearing interest at 7 per cent. per annum, in 4 equal payments?

46. The sum of $200 is to be applied in part towards the payment of a debt of $300, and in part to paying the interest, at 6% *in advance*, for 12 months, on the remainder of the debt. What is the amount of the payment that can be made on the debt?

Suggestion.—Let x represent the payment; then $(300-x) \times \frac{6}{100}$ is the interest on the remainder of the debt; and we have therefore the equation, $x + (300-x) \times \frac{6}{100} = 200$.

Ans., $193.62.

47. A is indebted to B $1,000, and is able to raise but $600. With this sum A proposes to pay a part of the debt, and the interest, at 8% in advance, on his note at 2 years for the remainder. For what sum should the note be drawn?

Ans., $476.19.

Partnership

SECTION III.

[**Note.**—For definitions, see Practical Arithmetic.]

91. **Principle.**—*In Simple Partnership, i. e., when all the capital is employed the same length of time, the fundamental principle is that the shares of the gains shall bear the same ratio as the shares of the stock.*

EXAMPLES.

Ex. **1.** A and B enter into partnership. A puts in $1,200 and B $1,800. They gain $900. What is each one's share of the gain?

Solution.—Let x and x_1 be their respective shares of the gain.

Then $x+x_1=900$
and $x:x_1::1200:1800::2:3$.
Whence $x+x_1:x::5:2$
or $900:x::5:2$
$\therefore\ x=360$, and $x_1=540$.

2. A, B, and C enter into partnership. A puts in $340, B $460, and C $500. They gain $390. What is the gain of each? *Ans.*, A's $102, B's $138, and C's $150

Operation. $x+x_1+x_2=390$. $x:x_1:x_2::340:460:500$, or $x+x_1+x_2:x::1300:340$, or $390:x::130:34$, or $3:x::1:34$. $\therefore\ x=102$. $x+x_1+x_2:x_1::1300:460$, or $390:x_1::130:46$. or $3:x_1::1:46$. $\therefore\ x_1=138$. $x+x_1+x_2:x_2::1300:500$, or $390:x_2::130:50$, or $3:x_2::1:50$. $\therefore\ x_2=150$.

3. Any number of persons as A, B, C, D, etc., unite in a partnership, putting in respectively a, b, c, d, etc., dollars each. The gain or loss is s. What portion falls to each?

Solution.—Let x_a, x_b, x_c, x_d, etc., be the respective shares of the gain or loss. Then $x_a + x_b + x_c + x_d +$, etc. $= s$,

and $x_a : x_b : x_c : x_d + - - -$ etc. $:: a : b : c : d - - -$ etc.

Whence $x_a + x_b + x_c + x_d +$ etc. $: x_a :: a + b + c + d +$ etc. $: a$

or $s : x_a :: a + b + c + d +$, etc. $: a$.

$$\therefore\ x_a = \frac{as}{a+b+c+d+,\ \text{etc.}}.$$

In like manner

$$x_b = \frac{bs}{a+b+c+d+,\ \text{etc.}},$$

$$x_c = \frac{cs}{a+b+c+d+,\ \text{etc.}},$$

$$x_d = \frac{ds}{a+b+c+d+,\ \text{etc.}}.$$

Scholium.—This is the common rule for Simple Partnership, viz.: Multiply the gain or loss by each partner's stock and divide the products by the whole stock.

[**Note.**—In such examples the solution is so simple that the equation scarcely renders any assistance. In the following its advantages will appear.]

4. Two men commenced trade together. The first put in \$40 more than the second; and the stock of the first was to that of the second as 5 to 4. What was the stock of each?

Ans., \$200, and \$160.

Operation.—Let $x =$ what the first put in. Then $x - 40 =$ what the second put in. And we have $x : x - 40 :: 5 : 4$, or $x : 40 :: 5 : 1$. $\therefore x = 200$.

5. Three men trading in company gained \$780, which was to be divided in proportion to their stock. A's stock was to B's as 2 to 3, and A's to C's as 2 to 5. What part of the gain should each have received?

Ans., A, \$156; B, \$234; C, \$390.

6. Three men trading in company, put in money in the following proportion: the first, 3 dollars as often as the second 7, and the third 5. They gain \$960. What is each man's share of the gain?

Ans., \$192; \$448; \$320.

7. A, B, and C found a purse of money; and it was mutually agreed that A should receive \$15 less than one half, that B should have \$13 more than one quarter, and that C should have the remainder, which was \$27. How many dollars did the purse contain? *Ans.*, \$100.

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92. Principle.—*In Compound Partnership,* i. e., *partnership in which the several partners' shares of the capital are in for different lengths of time, the gain or loss is divided in the ratio of the products of the several amounts of stock into the time which they respectively remained in the business. This is assuming that the use of* \$a *for time* t *in business is equal to* \$at *for time* 1.

8. A and B enter into partnership. A furnishes \$240 for 8 months, and B \$560 for 5 months. They lose \$118. How much does each man lose? *Ans.*, A 48, and B \$70.

Solution.—Let x_a = A's share of the loss, and x_b, B's. Then $x_a + x_b = 118$, and $x_a : x_b :: 8 \times 240 : 5 \times 560 :: 24 : 35$. Whence $x_a + x_b : x_a :: 59 : 24$, or $118 : x_a :: 59 : 24$, or $2 : x_a :: 2 : 48$. $\therefore\ x_a = 48$, and $x_b = 118 - 48 = 70$.

9. A, B, and C entered into partnership. A put in \$100 for 4 months, B \$300 for 2 months, and C \$500 for 3 months. They gained \$250. How much was each man's gain?

Operation. $x_a + x_b + x_c = 250$, and
$x_a : x_b : x_c :: 4 \times 100 : 2 \times 300 : 3 \times 500 :: 4 : 6 : 15$. Whence
$250 : x_a :: 25 : 4$, or $10 : x_a :: 1 : 4$. $\therefore\ x_a = 40$.
$250 : x_b :: 25 : 6$, or $10 : x_b :: 1 : 6$. $\therefore\ x_b = 60$.
$250 : x_c :: 25 : 15$, or $10 : x_c :: 1 : 15$. $\therefore\ x_c = 150$.

10. A, B, and C hire a pasture for \$180. A puts in 8 cows for 10 weeks, B 20 for 5 weeks, and C 30 for 9 weeks. How much ought each to pay?

Ans., A \$32, B \$40, and C \$108.

11. To gather a field of wheat, A furnished 8 laborers for 5 days, B 12 for 3 days, and C 6 for 4 days. For the whole work A, B, and C received \$45.50. How much should each have received? *Ans.*, A \$18.20, B \$16.38, and C \$10.92.

12. Any number of persons, as A, B, C, D, etc., unite in partnership, A putting in \$$a$ for time t_a; B, \$$b$ for time t_b; C, \$$c$ for time t_c; D, \$$d$ for time t_d, etc. The gain or loss is s. How is it to be shared by the partners?

Solution.—Letting x_a, x_b, x_c, x_d, etc., represent the respective shares of the gain or loss, we have

$$x_a + x_b + x_c + x_d +, \text{ etc.}, = s, \text{ and}$$

$$x_a : x_b : x_c : x_d \text{ - - - etc.}, :: at_a : bt_b : ct_c : dt_d \text{ - - - etc.} \quad \text{Whence}$$

$$s : x_a :: at_a + bt_b + ct_c + dt_d \text{ - - etc.}, : at_a. \quad \therefore\ x_a = \frac{at_a}{at_a + bt_b + ct_c + dt_d \text{ - - etc.}} s.$$

$$s : x_b :: at_a + bt_b + ct_c + dt_d \text{ - - etc.} : bt_b. \quad \therefore\ x_b = \frac{bt_b}{at_a + bt_b + ct_c + dt_d \text{ - - etc.}} s.$$

And in like manner for the others.

Scholium.—This is the common rule for Compound Partnership, viz.: Take the product of each partner's share of the stock into its time in trade. For any partner's share of the gain or loss multiply the whole gain or loss by the product of his stock into its time, and divide by the sum of the several shares of the stock into their respective times.

Alligation

SECTION IV.

Ex. **1.** A farmer mixes together 10 bushels of oats, at 40 cents a bushel, 15 bushels of corn at 50 cents a bushel, and 25 bushels of rye at 70 cents a bushel? What is the value of a bushel of the mixture? *Ans.*, 58 cents.

Solution.—Let x be the value of a bushel of the mixture of which there are $10+15+25=50$ bushels. Hence $50x$ represents the value of the whole mixture, and $50x=10\times 40+15\times 50+25\times 70=2900$. $\therefore\ x=58$.

2. A grocer mixes 120 pounds of sugar at 5 cents a pound, 150 pounds at 6 cents, and 130 pounds at 10 cents. What is the value of a pound of the mixture? *Ans.*, \$0.07.

3. A liquor dealer mixes 8 gallons of alcohol 100%, 12 gallons 80%, 25 gallons 60%, 40 gallons 40%, and 60 gallons 20% strong. What is the strength of the mixture?

Ans., $41\frac{23}{29}\%$.

4. One kind of wine is 40 cents a quart, and another 24. How much of each must be taken to make a quart worth 28 cents?

Statement. $x+y=1$. $40x+24y=28$.

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5. Three kinds of sugar are worth respectively 6, 8, and 10 cents a pound. How much must be taken of each to make a mixture worth 7 cents a pound?

Solution.—Let x, y, and z be the amounts of each required. x pounds at 6 cents a pound are $6x$; y at 8, $8y$; z at 10, $10z$. The whole amount is $x+y+z$, and the whole value $6x+8y+10z$. Hence $6x+8y+10z=7(x+y+z)$. But here are *three* unknown quantities, and the example gives but one set of conditions. The problem is therefore indeterminate; that is, there are not conditions enough given to *fix* the values of the unknown quantities.

Suppose we add the two conditions: To make a mixture of 48 lbs.; and that twice as much of the 6 cent sugar shall be used as of both the others. We then have the two additional equations $x+y+z=48$, and $x=2(y+z)$. These equations, together with the former, $6x+8y+10z=7(x+y+z)$ readily give $x=32$, $y=8$, and $z=8$.

93. Scholium.—The last example is a case in *Alligation Alternate*, as it is denominated in our Arithmetics. Such examples, as they are usually stated, are simply problems in which there are more unknown quantities than equations, and are hence *Indetermi-*

nate. The Indeterminate Analysis can not be treated in this volume, but a few further illustrations of such examples will be given. All the various methods of treating Alligation Alternate usually given in our Arithmetics are exceedingly cumbrous and perplexing to the pupil, and, after all, fail to give a full view of it. The propriety of puzzling pupils with any of them is exceedingly questionable. They are very clumsy and incomplete efforts at doing a thing which becomes very simple when the proper principles are developed, which principles cannot be brought forward in Common Arithmetic.

6. How much of each sort of grain, at 48, 50, and 68 cents a bushel, must be mixed together, so that the compound will be worth 60 cents a bushel?

Statement. x, y, and z being the amounts of each kind respectively, we have $48x+50y+68z = 60(x+y+z)$, or $6x+5y = 4z$. Now *any* real, positive values may be assigned to either two of these unknown quantities, which will give a positive value for the other.

Thus if $z = 4$, and $y = 2$, $6x = 16-10$, or $x = 1$. Verify these results.

Again, if $z = 5$, and $y = 1$, $6x = 20-5$, or $x = 2\frac{1}{2}$. Verify these results.

Again, if $z = 6$, and $y = 3$, $6x = 24-15$, or $x = 1\frac{1}{2}$. Verify these results.

In like manner an unlimited number of sets of answers can be obtained.

If, however, we try $z = 2$, and $y = 3$, we have $x = -1\frac{1}{6}$. The negative sign in this case shows an impossibility. The cause of this is evident when we notice that 2 bushels, at 68 cents, and 3 at 50, make 5 bushels, worth 286 cents, or $57\frac{1}{5}$ cents per bushel. This mixture cannot be made worth 60 cents per bushel by putting in grain worth only 48 cents.

The $-1\frac{1}{6}$ of the 48 cent grain indicated by this result, may be understood to mean that we are to *take out* of the 5 bushels, worth 286 cents, $1\frac{1}{6}$ bushel, worth 48 cents per bushel. This leaves $3\frac{5}{6}$ bushels, worth 230 cents, or just 60 cents per bushel.

7. A merchant has two kinds of wine. The first kind is worth 12 shillings per gallon, and the second is worth 7 shillings per gallon. How many gallons of each kind must he use in order to form a mixture worth 9 shillings per gallon?

Ans., Letting x and y represent the quantities respectively, he may take any quantity he pleases of either, so that he takes such an amount of the other as to preserve the relation $3x = 2y$. That is, he must take $1\frac{1}{2}$ times as much of the latter as of the former. If he takes 2 of the former, he must take 3 of the latter. If he takes 6 of the former, he must take 9 of the latter, etc.

8. How much corn at 48 cents, barley at 36 cents, and oats at 24 cents per bushel, must be taken to make a compound worth 30 cents per bushel?

Suggestion.—Show that he can take any amount he pleases of either one, if he takes proper amounts of the other two respectively. Show what he may take of the second and third kinds if he take 1 bushel of the first. Is there any limit to the number of ways he can make up the mixture when he takes 1 bushel of the first kind? Can he take 3 bushels of the first kind and 2 of the second? What value would this give to z (representing the amount of the third kind)?

9. A merchant wishes to mix 32 pounds of tea at 36 cents per pound, with some at 48 cents, and some at 72 cents. How many pounds of each kind must he take to form a mixture worth 56 cents per pound?

Suggestion.—The relation is $2y - x = 80$. Any value for either x or y may be taken which gives a positive value for the other; and *any* positive value of x will give in this case a positive value for the y, for $2y = 80 + x$. If $x = 4$, $y = 42$. If $x = 10$, $y = 45$. If $x = 1$, $y = 40\frac{1}{2}$, etc.

If we add to this example the condition that the whole amount of the mixture shall be 102 pounds, the problem becomes determinate; as there are then two equations with two unknown quantities. The equations, when reduced, are $2y - x = 80$, and $y + x = 70$. Whence $x = 20$, and $y = 50$.

10. A man bought horses at $50 each, oxen at $40, cows at $25, calves at $10, so that the average price per head was $30. How many were bought of each?

Suggestions.—The conditions to be met are $4x + 2y = z + 4w$, negative and fractional values being excluded by the nature of the problem. Can the conditions be met by taking 3 cows and 5 calves? Why? Can they by taking 2 cows and 5 calves? How? Can they by taking 4 horses and 6 cows? How?

11. A bought 240 barrels of molasses for \$4,320; worth, respectively, \$10, \$14, \$20, and \$22; how many barrels of each did he buy? *Ans.*, 40, 20, 160, and 20.

Suggestion.—The conditions are $x+y+z+w=240$, and $4x+2y=z+2w$. Here are two equations with four unknown quantities, hence any other *two* conditions may be imposed which do not conflict with the nature of the example. Can $x=60$ and $y=20$? Yes. This reduces the equations to $z+w=160$, and $z+2w=280$; from which $w=120$, and $z=40$. Can the condition that half the quantity shall be of the first two kinds be met? No. This gives $z+w=120$, which subtracted from $z+2w=4x+2y$, makes $w=2(2x+y)-120$. But since $x+y=120$, $2(2x+y)-120>120$. $\therefore\ w>120$, which would make z negative.

12. I have two kinds of molasses which cost me 20 and 30 cents per gallon; I wish to fill a hogshead, that will hold 80 gallons, with these two kinds. How much of each kind must be taken, that I may sell a gallon of the mixture at 25 cents per gallon and make 10 per cent. on my purchase?

Ans., $58\frac{2}{11}$ of 20 cents, and $21\frac{9}{11}$ of 30 cents.

13. A lumber merchant has several qualities of boards; and it is required to ascertain how many, at \$10 and \$15 per thousand feet, each, shall be sold on an order for 60 thousand feet, that the price for both qualities shall be \$12 per thousand feet. *Ans.*, 36 thousand at \$10, and 24 thousand at \$15.

14. How many ounces of gold 23 carats fine, and how many 20 carats fine, must be compounded with 8 ounces 18 carats fine, that the alloy of the three different qualities may be 22 carats fine? *Ans.*, 48 oz. of the 1st, and 8 oz. of the 2d.

[**Note.**—These applications might be extended to much greater length, did space permit. The equation renders important aid in many problems in Compound Interest, but their discussion usually requires a knowledge of Quadratics, and some of them of Logarithms. They must, therefore, be reserved for the future.]

QUADRATIC EQUATIONS

CHAPTER IV.

SECTION I.

PURE QUADRATICS.

94. **A Quadratic Equation** is an Equation of the second degree (**6, 8**).

95. Quadratic Equations are distinguished as *Pure* (called also *Incomplete*), and *Affected* (called also *Complete.*)

96. **A Pure Quadratic Equation** is an equation which contains no power of the unknown quantity but the second; as $ax^2+b=cd$, $x^2-3b=102$.

97. **An Affected Quadratic Equation** is an equation which contains terms of the second degree and also of the first, with respect to the unknown quantity or quantities; as $x^2-4x=12$, $5xy-x-y^2=16a$, $mxy+y=b$.

98. **A Root** of an equation is a quantity which substituted for the unknown quantity satisfies the equation.

Note the difference between this use of the word *root*, and its former use as defined in **Art. 39.** Here we speak of "the root of an equation," meaning the value of the unknown quantity; in the former sense we speak of "the root of a number," meaning one of its equal factors.

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99. **Prob.—To solve a Pure Quadratic Equation.**

Rule.—*I. Transpose all the terms containing the unknown quantity into the first member, and unite them into one, clearing of fractions if necessary.*

II. Transpose the known terms into the second member.

III. Divide by the coefficient of the unknown quantity, and extract the square root of each member.

Demonstration.—According to the definition of a Pure Quadratic, all the terms containing the unknown quantity contain its square. Hence they can be transposed and united into one by adding with reference to the square of the unknown quantity (**16**). Extracting the square root of the first member gives the first power of the unknown quantity, *i. e.*, the quantity itself. And taking the square root of each member does not destroy the equation, since like roots of equal quantities are equal.

100. **Cor. 1.**—*Every Pure Quadratic Equation has two roots numerically equal but with opposite signs.*

This is apparent since every such equation, as the process of solution shows, can be reduced to the form $x^2 = a$ (a representing any quantity whatever). Whence, extracting the root, we have $x = \pm\sqrt{a}$; as the square root of a quantity is both + and — (**203**).

Scholium.—The question naturally arises, Why not put the ambiguous sign (the ±) before the x, as well as before the second member? It is proper to; but there is no advantage gained by it. Thus, if we write $\pm x = \pm\sqrt{a}$, we have $+x = \pm\sqrt{a}$, or $-x = \pm\sqrt{a}$. But the former is $x = \pm\sqrt{a}$, and the latter becomes so by changing the signs of both members. So that all we learn in either case, is that $x = +\sqrt{a}$, and $x = -\sqrt{a}$.

101. **Cor. 2.**—*The roots of a Pure Quadratic Equation may both be imaginary, and* BOTH *will be if* ONE *is.* For if after having transposed and reduced to the form $x^2 = a$, the second member is negative, as $x^2 = -a$, extracting the square root gives $x = +\sqrt{-a}$, and $x = -\sqrt{-a}$, both imaginary.

EXAMPLES.

Ex. 1. Given $3x^2 - 10 - x^2 = 12 + 4x^2 - 54$ to find the value of x.

Model Solution.

Operation. $3x^2 - 10 - x^2 = 12 + 4x^2 - 54$,

(2) $-2x^2 = -32$,

(3) $x^2 = 16$, $x = \pm 4$.

Explanation.—Transposing and uniting terms, I have $-2x^2=-32$. Dividing each member by -2, I have $x^2=16$. Extracting the square root of each member, I find $x=\pm 4$ (read, "x equals plus and minus 4").

Verification.—Substituting $+4$ for x, the equation becomes $48-10-16=12+64-54$, or $22=22$. Substituting -4 gives just the same since -4 squared, $(-4)^2$, is the same as $+4$ squared.

2. Given $x^2+1=\frac{x^2}{4}+4$, to find x. *Roots*, $x=\pm 2$.

3. Given $\frac{x(9+2x)}{15}=\frac{3x+6}{5}$, to find x.

Roots, $x=\pm 3$.

4. Given $\frac{5}{4+x}+\frac{5}{4-x}=\frac{8}{3}$, to find the values of x.

Roots, $x=\pm 1$.

5. Given $x^2-ab=d$, to find the values of x.

Roots, $x=\pm\sqrt{d+ab}$.

6. Given $\frac{x^2}{2}-3+\frac{5x^2}{12}=\frac{7}{24}-x^2+\frac{335}{24}$, to find the values of x. *Roots*, $x=\pm 3$.

7. Given $13-\sqrt{3x^2+16}=5$, to find the values of x.

Roots, $x=\pm 4$.

8. Given $x+\sqrt{x^2+a}=\frac{2a}{\sqrt{x^2+a}}$, to find the values of x.

Roots, $x=\pm\frac{1}{3}\sqrt{3a}$.

9. Given $\frac{15}{\sqrt{x^2+5}}=x+\sqrt{x^2+5}$. *Roots*, $x=\pm 2$.

10. Given $\sqrt{x+a}=\frac{a+b}{\sqrt{x-a}}$. *Roots*, $x=\sqrt{2a(a+b)+b^2}$.

11. Given $\frac{x}{a}-\frac{a}{x}=ax.$ *Roots*, $x=\pm\frac{a}{\sqrt{1-a^2}}.$

12. Given $\frac{m}{x}-x=\frac{2\sqrt{mn}}{x}-\frac{n}{x}.$

Roots, $x=\pm(\sqrt{m}-\sqrt{n}).$

13. Given $\frac{x^2}{b^2}-\frac{a^2}{b^2}+1=\frac{x^2}{a^2}-1+\frac{b^2}{a^2}.$

Roots, $x=\pm\sqrt{a^2-b^2}.$

14. Given $12-x^2:\frac{1}{2}x^2::100:25.$ *Roots*, $x=\pm 2.$

Operation. $12-x^2:\frac{1}{2}x^2::100:25$
$12-x^2:x^2::50:25::2:1$
$12:x^2::3:1$
$4:x^2::1:1 \quad \therefore x=\pm 2.$

[**Note.**—Use the principles of proportion in solving these.]

15. Given $\frac{1}{2}x^2+\frac{1}{4}x^2-3:\frac{1}{4}x^2-\frac{1}{8}x^2+3::9:3.$

Roots, $x=\pm 4\sqrt{2}.$

16. Given $\frac{1}{2}(x^2-5)^2:x^2-5::2:1.$ *Roots*, $x=\pm 3.$

Operation. $\frac{1}{2}(x^2-5)^2:x^2-5::2:1$
$x^2-5:1::4:1$
$x^2-5:5::4:5$
$x^2:5::9:5 \quad \therefore x=\pm 3.$

17. Given $\frac{1}{3}(11+x^2):\frac{1}{7}(4x^2-2)::5:2.$

Roots, $x=\pm 2.$

18. Given $x^2+4:x^2-11::100:40.$

Roots, $x=\pm\sqrt{21}.$

19. Given $\frac{x^3+x+8}{x^3+4}=2-\frac{x^3+x-8}{x^3-4}.$ *Roots*, $x=\pm 2.$

20. Given $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}.$ *Roots*, $x=\pm 8.$

21. Given $(x+2)^2=4x+5.$ *Roots*, $x=\pm 1.$

22. Given $\frac{1}{3}(x^2-12)=\frac{1}{4}x^2-1.$ *Roots*, $x=\pm 6.$

23. Given $\frac{x+7}{x^2-7x}-\frac{x-7}{x^2+7x}=\frac{7}{x^2-73}.$

Roots, $x=\pm 9.$

24. Given $x+\sqrt{x^2-\sqrt{2-4x}}=1$. *Roots*, $x=\pm\frac{1}{2}$.

25. Given $\sqrt{a+x}=\sqrt{x+\sqrt{x^2+b^2}}$.

Roots, $x=\pm\sqrt{a^2-b^2}$.

26. Given $\sqrt{1+x^2}+\sqrt{1+x^2+\sqrt{1-x^2}}=\sqrt{1-x^2}$.

Verify. *Roots*, $x=\pm\frac{2}{5}\sqrt{-6}$.

27. Given $\sqrt[4]{b^4+x^4}=\sqrt{a^2+x^2}$.

Roots, $x=\pm\dfrac{1}{2a}\sqrt{2(b^4-a^4)}$.

28. Given $b-\sqrt{a^2+x^2}=\dfrac{a^2-x^2}{a+\sqrt{a^2+x^2}}$.

Roots, $x=\pm\dfrac{a^2}{a-b}\sqrt{3-\dfrac{2b}{a}}$.

29. Given $\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}=b$. *Roots*, $x=\pm\dfrac{2a\sqrt{b}}{1-b}$.

30. Given $\dfrac{\sqrt{3x^2+4}+2}{\sqrt{3x^2+4}-2}=3$. *Roots*, $x=\pm 2$.

31. Given $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\frac{1}{2}$. *Roots*, $x=\pm\frac{1}{2}\sqrt{5}$.

32. Given $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$.

Roots, $x=\pm\sqrt{3}$.

33. Given $\dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}}=\dfrac{a}{x^2}$.

Roots, $x=\pm\frac{1}{2}a\sqrt{3}$.

34. Given $\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}=\dfrac{b}{c}$. *Roots*, $x=\pm\dfrac{a(b-c)}{2\sqrt{bc}}$.

35. Given $\dfrac{1}{\sqrt{1-x^2}+1}+\dfrac{1}{\sqrt{1-x^2}-1}=\dfrac{1}{x^2}$.

Roots, $x=\pm\frac{1}{2}\sqrt{3}$.

36. Given $\dfrac{1+x}{1+x+\sqrt{1+x^2}}=a-\dfrac{1-x}{1-x+\sqrt{1+x^2}}$.

Roots, $x=\pm\sqrt{(a-2)^2-1}$.

37. Given $\dfrac{1}{x+\sqrt{2-x^2}}+\dfrac{1}{x-\sqrt{2-x^2}}=ax$.

Roots, $x=\pm\sqrt{\dfrac{a+1}{a}}$.

38. Given $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}=a$.

Roots, $x=\pm\dfrac{4}{a}\sqrt{1-\dfrac{4}{a^2}}$.

APPLICATIONS.

Ex. 1. What two numbers are those whose sum is to the greater as 10 : 7, and whose sum multiplied by the less produces 270? *Ans.*, 21 and 9.

Suggestion.—Let $10x =$ the sum of the numbers, and $7x$ the greater.

Scholium.—It is customary to omit the negative roots in giving answers to examples, the nature of which renders such answers impossible. In this case the question is about pure number, and hence the answers should be given without signs.

2. There are two numbers whose ratio is that of 4 to 5, and the difference of whose squares is 81. What are the numbers? *Ans.*, 12 and 15.

3. What two numbers are those whose difference is to the greater as 2 to 9, and the difference of whose squares is 128? Verify.

4. Find three numbers which bear the same ratio to each other as $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ do to each other, and the sum of whose squares is 724. *Numbers,* 12, 16, 18.

5. Find three numbers in the ratio of m, n, and p, the sum of whose squares is equal to a.

Numbers, $\pm\sqrt{\frac{am^2}{m^2+n^2+p^2}}$, $\pm\sqrt{\frac{an^2}{m^2+n^2+p^2}}$, and $\pm\sqrt{\frac{ap^2}{m^2+n^2+p^2}}$.

6. Divide 14 into two parts so that the greater part divided by the less shall be to the less divided by the greater as 16 to 9.

Suggestion.—Having $\frac{x}{14-x} : \frac{14-x}{\blacksquare} :: 16 : 9$, it follows that $x^2 : (14-x)^2 :: 16 : 9$, and $x : 14-x :: 4 : 3$, and $x : 14 :: 4 : 7$. $\therefore x = 8$, and $14-x=6$.

7. Divide a into two parts so that the greater part divided by the less shall be to the less divided by the greater as m to n.

Parts, $\frac{a\sqrt{m}}{\sqrt{m}+\sqrt{n}}$, and $\frac{a\sqrt{n}}{\sqrt{m}+\sqrt{n}}$.

Scholium.—Example 7 is example 6 generalized. The pupil should deduce the results in the former from these. Thus, substituting $a=14$, $m=16$, and $n=9$, $\frac{a\sqrt{m}}{\sqrt{m}+\sqrt{n}}=8$, etc.

8. What two numbers are they, whose product is 126, and the quotient of the greater divided by the less, $3\frac{1}{2}$? Generalize this. *Ans.,* 6 and 21.

9. The sum of the squares of two numbers is 370, and the difference of their squares 208. Required the numbers. *Numbers,* 9 and 17.

10. Generalize the 9th, and show that $\frac{1}{2}\sqrt{2(s+d)}$ and $\frac{1}{2}\sqrt{2(s-d)}$ are general results.

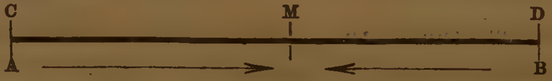
11. For comparatively small distances above the earth's surface the distances through which bodies fall under the influence of gravity are as the squares of the times. Thus, if one body is falling 2 seconds and another 3, the distances fallen through are as 4 : 9. A body falls 4 times as far in 2 seconds as in 1, and 9 times as far in 3 seconds. These facts are learned both by observation and theoretically. It is also observed that a body falls $16\frac{1}{12}$ feet in one second. How long is a body in falling 500 feet? One mile (5280 ft.)? Five miles? *Ans.*, To fall 500 ft. requires 5.58 seconds. To fall 5 miles requires 40.51 seconds.

12. A and B lay out some money in a speculation. A disposes of his bargain for \$11, and gains as much per cent. as B lays out. B succeeds in gaining \$36; and it appears that A gains four times as much per cent. as B. Required the capital of each. *Results*, \$5 = A's capital, and \$120 = B's.

13. A money safe contains a certain number of drawers. In each drawer there are as many divisions as there are drawers, and in each division there are four times as many dollars as there are drawers. The whole sum in the safe is \$5,324; what is the number of drawers? *Ans.*, 11.

14. Two travelers, A and B, set out to meet each other; A leaving the town C at the same time that B left D. They travelled the direct road from C to D, and on meeting it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What is the distance between C and D? *Ans.*, 126 miles.

Sug.



If $x = CM =$ the distance A travelled, then $x-18 = MD =$ the distance B travelled. $\frac{x-18}{15\frac{3}{4}}$ = distance A travelled a day; and $\frac{x}{28}$ = distance B travelled a day. *Notice that the times are equal.*

15. From two places at an unknown distance, two bodies, A and B, move toward each other till they meet, A going a miles more than B. A would have described B's distance in n hours, and B would have described A's distance in m hours. What was the distance of the two places from each other? *Ans.*, $a \times \frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}$.

16. A and B engaged to work for a certain number of days. A lost 4 days of the time and received $18.75. B lost 7 days and received $12. Now had A lost 7 and B 4 days, the amounts received would have been equal. How long did they engage to work and at what rates?

Ans., Whole time, 19 days.

Suggestion.—If x = the whole time, what represents A's daily wages? What B's? After the equation is formed, see if you cannot strike out a numerical factor from both members, and extract the root *without* expanding.

17. A vintner drew a certain quantity of wine out of a full vessel that held 256 gallons; and then filled the vessel with water, and drew off the same number of gallons as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time? *Ans.*, 64, 48, 36, and 27 gallons.

Suggestion.—If he drew out $\frac{1}{x}$ part of the contents of the cask each time, there remained after the first drawing $\frac{x-1}{x}$th of the wine; after the second $\frac{x-1}{x} \times \frac{x-1}{x}$ or $\frac{(x-1)^2}{x^2}$, and after the fourth $\frac{(x-1)^4}{x^4}$. $\therefore \frac{(x-1)^4}{x^4} = \frac{81}{256}$, or $\frac{x-1}{x} = \frac{3}{4}$. Whence $\frac{1}{x} = \frac{1}{4}$.

18. A number a is diminished by the nth part of itself, this remainder is diminished by the nth part of itself, and so on to the fourth remainder, which is equal to b. Required the value of n.

Ans., $\frac{\sqrt[4]{a}}{\sqrt[4]{a}-\sqrt[4]{b}}$.

19. There is a number such that, if the square root of three times its square + 4, be taken, the quotient of this root increased by 2, divided by the root diminished by 2, is 3. What is the number?

Query.—Which of the equations in the preceding part of this section does this give rise to?

20. If the square root of the difference between the square of a certain number and 2, be both added to and subtracted from the number itself, the sum of the reciprocals of the result is $\frac{1}{15}$ of the number itself. What is the number?

Query.—Which of the equations in the preceding part of this section does this give rise to? With what modifications?

QUADRATIC EQUATIONS

AFFECTED.

SECTION II.

102. **An Affected Quadratic** equation is an equation which contains terms of the second degree and also of the first with respect to the unknown quantity.

$x^2-3x=12$, $4x+3ax^2=\frac{2ax+3bx^2}{5}$, and $\frac{a^2x^2}{b^2}-4ax+3b^2=0$, are affected quadratic equations.

103. Prob.—To solve an Affected Quadratic Equation.

Rule.—*I. Reduce the equation to the form* $x^2+ax=b$.*

II. Add the square of half the coefficients of the second term to each member of the equation.

III. Extract the square root of each member, thus producing a simple equation from which the value of the unknown quantity is found by simple transposition.

Demonstration.—By definition an Affected Quadratic Equation contains but three kinds of terms, viz.: terms containing the square of the unknown quantity, terms containing the first power of the unknown quantity, and *known* terms. Each of the three kinds of terms may, by clearing of fractions, transposition, and uniting, as the particular example may require, be united into one, and the results arranged in the order given. If, then, the first term, *i. e.* the one containing the square of the unknown quantity, has a coefficient other than unity, or is negative, its coefficient can be rendered unity and positive without destroying the equation by dividing both the members by whatever coefficient this term may have after the first reductions. The equation will then take the form $x^2 \pm ax = \pm b$. Now adding $\left(\frac{a}{2}\right)^2$ to the first member makes it a perfect square $\left(\text{the square of } x \pm \frac{a}{2}\right)$, since a trinomial is a perfect square when one of its terms (the middle one, ax, in this case) is ± twice the product of the square roots of the other two, these two being both positive (**123,** PART I). But if we add the square of half the coefficient of the second term to the first member to make it a complete square, we must add it to the second member to preserve the equality of the members. Having extracted the square root of each member, these roots are equal, since like roots of equals are equal. Now, since the first term of the trinomial square is x^2, and the last $\left(\frac{a^2}{4}\right)$ does not contain x, its square root is a binomial consisting of $x \pm$ the square root of its third term, or half the coefficient of the middle term, and hence a known quantity. The square root of the second member can be taken exactly, approximately, or indicated, as the case may be. Finally, as the first term of this resulting equation is simply the unknown quantity, its value is found by transposing the second term.

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* The characteristics of this form are, that the first member consists of two terms, the first of which is positive and contains simply the square of the unknown quantity, its coefficient being unity, while the second has the first power of the unknown quantity, with any coefficient (a) positive or negative, integral or fractional; and the second member consists of known terms (b).

EXAMPLES.

Ex. 1. Given $6-x-\frac{x^2}{8}=x-2\frac{1}{4}+\frac{x^2}{8}$, to find the value of x, and verify.

Model Solution—Operation.

$$6-x-\frac{x^2}{8}=x-2\tfrac{1}{4}+\frac{x^2}{8}$$
$$48-8x-x^2=8x-18+x^2$$
$$-2x^2-16x=-66$$
$$x^2+8x=33$$
$$x^2+8x+16=33+16=49$$
$$x+4=\pm 7,$$
$$x=\pm 7-4=3, \text{ and } -11$$

Explanation.—Clearing the equation of fractions, transposing and uniting, and dividing each member by -2, I have $x^2+8x=33$. Now since $8x$ contains the square root of x^2 as one of its factors, the other factor, 8, is twice the square root of the other term of a trinomial square (**123,** PART I). Hence $\frac{1}{2}$ of 8 squared (16) is the third term. Adding this term I have $x^2+8x+16$, which is a perfect square. But as I have added 16 to the first member to make it a perfect square, I must add it to the second member to preserve the equality. This gives $x^2+8x+16=49$. Extracting the square root of both members, I have $x+4=\pm 7$. Finally, transposing the 4, I have $x=-4\pm 7$, *i. e.* $x=3$ and -11. Both are correct. Hence there are two roots (values of x) of this equation, 3 and -11.

Verification.—To verify the value $x=3$, I substitute 3 for x in the given equation, and have $6-3-\frac{9}{8}=3-2\frac{1}{4}+\frac{9}{8}$, or $3-\frac{9}{8}=\frac{3}{4}+\frac{9}{8}$, or $\frac{15}{8}=\frac{15}{8}$. To verify the value $x=-11$, I substitute for x, -11, in the given equation, and have $6+11-\frac{121}{8}=-11-2\frac{1}{4}+\frac{121}{8}$, or $17-\frac{121}{8}=-13\frac{1}{4}+\frac{121}{8}$, or $\frac{15}{8}=\frac{15}{8}$.

2. Given $x^2-8x+5=14$, to find the values of x, and verify. *Result,* $x=9$, and -1.

3. Find the roots of the equation $x^2-12x+30=3$, and verify. *Roots,* 9 and 3.

4. Find the roots of $x-2=\frac{4x-9}{x}$.

Suggestion.—This reduces to $x^2-6x=-9$. Whence, completing the square, $x^2-6x+9=9-9=0$, and $x-3=0$, or $x=3$. In this case it appears that the equation has but one root.

5. Given $\frac{x}{5}=\frac{2x-5}{x}$, to find the values of x.

6. Find the roots of $3(x-4)=\frac{12(x-5)}{x}$.

Suggestion.—This reduces to $x^2-8x=-20$. Whence, completing the square, $x^2-8x+16=16-20=-4$, and extracting the root, $x-4=\pm 2\sqrt{-1}$, or $x=4\pm 2\sqrt{-1}$, two imaginary roots.

7. Find the values of x in $\frac{x}{5}=\frac{x-9}{x-5}$.

Results, $x=5+2\sqrt{-5}$, and $5-2\sqrt{-5}$.

8. Find the roots of $x+\frac{300}{x}+73=3\frac{x-300}{x}$.

Roots, -30, and -40.

9. Find the roots of $\frac{4(2x+6)}{2x}=\frac{8}{x}-x$.

Only one root, -2.

10. Find the roots of $7(x+7)+\frac{7(3x+50)}{x}=0$.

Roots, $5(-1+\sqrt{-1})$, and $5(-1-\sqrt{-1})$, or as the same may be written, $-5(1-\sqrt{-1})$, and $-5(1+\sqrt{-1})$.

11. Find the roots of $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$, and verify.
Roots, 7, and $-6\frac{1}{3}$.

Scholium.—This process of adding the square of half the coefficient of the first power of the unknown quantity to the first member, in order to make it a perfect square, is called COMPLETING THE SQUARE. There are a variety of other ways of completing the square of an affected quadratic, some of which will be given as we proceed; but this is the most important. This method will solve all cases: others are mere matters of convenience, in special cases.

12. Given $x^2-x+3=45$, to find its roots, and verify.

13. Given $2x^2+8x-20=70$, to find its roots, and verify.

14. Given $5-\frac{10x}{3}=13\frac{1}{3}x-\frac{5x^2-65}{3}$, to find its roots, and verify.

15. Given $6x+\frac{35-3x}{x}=44$, to find the values of x.

Suggestion.—Clearing of fractions, $6x^2+35-3x=44x$,
Transposing and uniting, $6x^2-47x=-35$,
Dividing by 6, $x^2-\frac{47}{6}x=-\frac{35}{6}$,
Completing the square, $x^2-\frac{47}{6}x+(\frac{47}{12})^2=(\frac{47}{12})^2-\frac{35}{6}=\frac{1369}{144}$,
Extracting the root, $x-\frac{47}{12}=\pm\frac{37}{12}$,
Transposing, $x=\frac{47}{12}\pm\frac{37}{12}=7$, and $\frac{5}{6}$.

16. Find the roots of $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$.

Suggestion.—Notice the compound negative term. Cleared of fractions and reduced, the equation becomes $x^2-3x=4$. $\therefore x=4$, and -1.

17. Find the roots of $\frac{16}{x}-\frac{100-9x}{4x^2}=3$.

Suggestion.—Multiply by $4x^2$ to clear of fractions.

18. Find the roots of $3x^2-20x-62 = 7x-2x^2+100$.

Roots, 9, and $-3\frac{3}{5}$.

19. Find the roots of $2x-2 = 2+\frac{6}{x}$.

Roots, 3, and -1.

20. Find the roots of $\frac{1}{2}x^2-\frac{1}{3}x+7 = 8-\frac{3}{8}$.

Roots, $\frac{3}{2}$, and $-\frac{5}{6}$.

21. Find the values of x in the equation $\frac{48}{x+3}+5 = \frac{165}{x+10}$.

Result, $x = 5\frac{2}{5}$, and 5.

22. Find the values of x in the equation $\frac{8x}{x+2}-\frac{20}{3x} = 6$.

Result, $x = 10$, and $-\frac{2}{3}$.

23. Given $\frac{1}{3}(x+4)-\frac{7-x}{x-3} = \frac{1}{9}(4x+7)-1$, to find the values of x.

Result, $x = 21$, and 5.

24. Given $\frac{x+12}{x}+\frac{x}{x+12} = 5\frac{3}{15}$, to find the roots, and verify.

25. Find the roots of $\frac{1}{2}(8-x)-\frac{2x-11}{x-3} = \frac{1}{6}(x-2)$, and verify.

26. Find the roots of $\frac{2x+9}{9}+\frac{4x-3}{4x+3} = 3+\frac{3x-16}{18}$, and verify.

27. Find the values of x in the equation $3x^2-2ax = b$.

Result, $x = \frac{a\pm\sqrt{a^2+3b}}{3}$.

Operation. $3x^2-2ax=b;$

Dividing by 3, $x^2-\frac{2a}{3}x=\frac{b}{3},$

Completing the square, $x^2-\frac{2a}{3}x+\left(\frac{a}{3}\right)^2=\frac{a^2}{9}+\frac{b}{3}=\frac{a^2+3b}{9},$

Extracting the square root, $x-\frac{a}{3}=\pm\frac{1}{3}\sqrt{a^2+3b},$

Transposing, $x=\frac{a}{3}\pm\frac{\sqrt{a^2+3b}}{3}$, or $\frac{a\pm\sqrt{a^2+3b}}{3}.$

Scholium.—The form $\frac{a\pm\sqrt{a^2+3b}}{3}$ signifies that there are two values of x; *i. e.*, that each of the signs + and − may be used. Thus the values in this case are

$$x=\frac{a+\sqrt{a^2+3b}}{3},\ \text{and}\ x=\frac{a-\sqrt{a^2+3b}}{3}.$$

28. Find the roots of $3x^2+5ax=m$.

Roots, $\frac{-5a\pm\sqrt{25a^2+12m}}{6}.$

29. Find the roots of $9a^4b^4x^2-6a^3b^2x=b^2$.

Roots, $\frac{a\pm\sqrt{a^2+b^2}}{3a^2b^2}.$

30. Find the values of x in the equation $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}.$

Result, $x=1\pm\sqrt{1-a^2}.$

31. Find the roots of $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}.$

Roots, $\frac{3}{4}a$, and $\frac{1}{2}a$.

104. **Cor. 1.**—*An affected quadratic equation has two roots. These roots may both be positive, both be negative, or one positive and the other negative. They are both real, or both imaginary.*

Demonstration.—Let $x^2+px=q$ be any affected quadratic equation reduced to the form for completing the square. In this form p and q may be either positive or negative, integral or fractional.

Solving this equation we have $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4}+q}$. We will now observe what different forms this expression can take, depending upon the signs and relative values of p and q.

1st. *When* p *and* q *are both positive.* The *signs* will then stand as given; *i. e.*, $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4}+q}$. Now, it is evident that $\sqrt{\frac{p^2}{4}+q} > \frac{p}{2}$, for $\sqrt{\frac{p^2}{4}+q}$ is the square root of something *more* than $\frac{p^2}{4}$. Hence, as $\frac{p}{2} < \sqrt{\frac{p^2}{4}+q}$, $-\frac{p}{2} + \sqrt{\frac{p^2}{4}+q}$ is *positive*; but $-\frac{p}{2} - \sqrt{\frac{p^2}{4}+q}$ is *negative*, for both parts are negative. Moreover the negative root is numerically greater than the positive, since the former is the numerical sum of the two parts, and the latter the numerical difference. ∴ When p and q are both + in the given form, one root is positive and the other negative, and the negative root is numerically greater than the positive one. See Example 1, above.

2nd. *When* p *is negative and* q *positive.* We then have $x = -\frac{-p}{2} \pm \sqrt{\frac{(-p)^2}{4}+q} = \frac{p}{2} \pm \sqrt{\frac{p^2}{4}+q}$. If we take the plus sign of the radical, x is positive; but if we take the — sign, x is negative, since $\sqrt{\frac{p^2}{4}+q} > \frac{p}{2}$. Moreover, the positive root is numerically the greater. ∴ When p is negative and q positive, one root is positive and the other negative; but the positive root is numerically greater than the negative. See Example 2, above.

3rd. *When* p *and* q *are both negative.* We then have $x = -\frac{-p}{2} \pm \sqrt{\frac{(-p)^2}{4}+(-q)} = \frac{p}{2} \pm \sqrt{\frac{p^2}{4}-q}$. In this if $\frac{p^2}{4} > q$, $\sqrt{\frac{p^2}{4}-q}$ is real, and as it is less than $\frac{p}{2}$, both values are positive. See Ex. 3. If $\frac{p^2}{4} = q$, $\sqrt{\frac{p^2}{4}-q} = 0$, and there is but one value of x, and this is positive. (It is customary to *call* this *two equal* positive roots, for the sake of analogy, and for other reasons which cannot now be appreciated by the pupil.) See Examples 4 and 5. If $\frac{p^2}{4} < q$, $\sqrt{\frac{p^2}{4} - q}$ becomes the square root of a negative quantity and hence imaginary. See Examples 6 and 7.

4th. *When* p *is positive and* q *negative.* We then have $x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$. As before, this gives two *real* roots when $q < \frac{p^2}{4}$. When this is the case both roots are *negative.* [Let the pupil show how this is seen.] When $q = \frac{p^2}{4}$, the roots are equal and negative; *i. e.* there is but one. When $\frac{p^2}{4} < q$ both roots are imaginary. See Examples 8, 9, and 10.

[**Note.**—It is not important that the pupil remember all these forms; but it is an excellent exercise to give the discussion. The ingenious student can put the results in a very neat analytical table.]

Scholium.—It may be asked why, when we extract the square root of each member of the equation $x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}$, we write the ambiguous sign only before the root of the second member. The reason is the same as given under the Pure Quadratics, Art. 100, *Scholium.* Thus, in strict propriety, the square root of each member of this equation being taken or indicated gives $\pm\left(x + \frac{p}{2}\right) = \pm\sqrt{\frac{p^2}{4} + q}$. But take these signs in any order we can, it amounts to taking the roots as having like signs (both +, or both −) or unlike signs (one + and the other −). Hence it is sufficient to give the ambiguous sign to one member only; and it is most convenient to give it to the second.

32. Given $\sqrt{x+5} \times \sqrt{x+12} = 12$, to find the values of x.

Result, $x = 4$, and -21.

Suggestion.—First clear the equation of radicals.

33. Given $\sqrt{4x+5} \times \sqrt{7x+1} = 30$, to find the values of x.

Values, $x = 5$, and $-6\frac{11}{28}$.

34. Given $x + 5 = \sqrt{x+5} + 6$, to find the values of x.

Values, $x = 4$, and -1.

35. Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$, to find the values of x. *Values*, $x=9$, and -12.

Suggestion.—Put the equation in the form $x+6=3\sqrt{x+16}$, and then square.

36. Given $3\sqrt{x+6}+2=x+\sqrt{x+6}$, to find the values of x. *Result*, $x=10$, and -2.

37. Given $y+\frac{1}{y}=\frac{4}{\sqrt{3}}$, to find the roots.

Result, $x=\sqrt{3}$, $\frac{1}{3}\sqrt{3}$.

Suggestion.—Multiply by y and transpose, and $y^2-\frac{4}{\sqrt{3}}y=-1$. Completing the square, $y^2-\frac{4}{\sqrt{3}}y+\frac{4}{3}=\frac{4}{3}-1=\frac{1}{3}$. Extracting the root, $y-\frac{2}{\sqrt{3}}=\pm\sqrt{\frac{1}{3}}$. $\therefore y=\frac{2}{\sqrt{3}}\pm\sqrt{\frac{1}{3}}=2\sqrt{\frac{1}{3}}\pm\sqrt{\frac{1}{3}}=3\sqrt{\frac{1}{3}}$, and $\sqrt{\frac{1}{3}}=\sqrt{3}$, and $\frac{1}{3}\sqrt{3}$.

105. **Cor. 2.**—*An affected quadratic being reduced to the form* $x^2+px=q$, *the value of* x *is half the coefficient of the second term taken with the opposite sign,* ± *the square root of the sum of the square of this half coefficient, and the known term of the equation. This is observed directly from the form* $x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}$, *and more in detail in the demonstration of the preceding corollary.*

[**Note.**—The pupil should use this method in practice, but be careful that the complete method and its full demonstration is not lost sight of.]

38. Write out the roots of the following without going through the operations of completing the square, etc.; $z^2+4z=60$; $y^2-4y=60$; $x^2+16x=-60$; $x^2-16x=-60$.

39. Reduce the following to the form $x^2+px=q$, and then write out the roots as above:

$3x^2+2x+6=11$, gives $x^2+\frac{2}{3}x=\frac{5}{3}$. Whence $x=-\frac{1}{3}$ $\pm\sqrt{\frac{1}{9}+\frac{5}{3}}=-\frac{1}{3}\pm\frac{4}{3}=1$, and $-\frac{5}{3}$.

$\frac{9x^2}{16}+\frac{30x}{8}=-4$, gives $x^2+\frac{20}{3}x=-\frac{64}{9}$. Whence $x=$ $-\frac{10}{3}\pm\sqrt{\frac{100}{9}-\frac{64}{9}}=\frac{-10\pm 6}{3}=-1\frac{1}{3}$, and $-5\frac{1}{3}$.

$4x-\frac{14-x}{x+1}=14$, gives $x^2-\frac{9}{4}x=7$. Whence $x=\frac{9}{8}$ $\pm\sqrt{\frac{81}{64}+7}=\frac{9\pm 23}{8}=4$, and $-1\frac{3}{4}$.

40. Given $9x^{-2}-12x^{-1}=-3$, to find the values of x.

Values, $x=3$, and 1.

Suggestion. $9x^{-2}=\frac{9}{x^2}$

41. Given $\dfrac{y+\frac{1}{y}}{y-\frac{1}{y}}+\dfrac{1+\frac{1}{y}}{1-\frac{1}{y}}=\frac{13}{4}$, to find the roots.

Suggestion.—Reduce the complex fractions to simple ones by multiplying numerator and denominator by y. Whence $\frac{y^2+1}{y^2-1}+\frac{y+1}{y-1}$ $=\frac{13}{4}$. Multiply by $4(y^2-1)$, and $4y^2+4+4y^2+8y+4=13y^2-13$. $\therefore\ y=3$, and $-\frac{7}{5}$.

42. Given $\sqrt{a+x}+\sqrt{a-x}=\frac{12a}{5\sqrt{a+x}}$, to find x.

Result, $x=\frac{4a}{5}$, and $\frac{3a}{5}$.

43. Given $\sqrt{x+a}-\sqrt{x+b}=\sqrt{2x}$, to find the roots.

Result, $x=-\frac{a+b}{2}\pm\frac{1}{2}\sqrt{2a^2+2b^2}$.

Suggestion. $\sqrt{x+a}-\sqrt{2x}=\sqrt{x+b}$. $x+a-2\sqrt{2x^2+2ax}+2x$ $=x+b$. $-2\sqrt{2x^2+2ax}=b-a-2x$. $8x^2+8ax=b^2-2ab-4bx+a^2$ $+4ax+4x^2$.

44. Given $\frac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})x=\frac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}$, to find the values of x. *Result*, $x=a$, and $-b$.

Suggestion. $\frac{1}{(ab^3)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}=\frac{ab}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$. Whence the given equation becomes $x^2-(a-b)x=ab$.

45. Given $\frac{x}{\sqrt{x}+\sqrt{a-x}}+\frac{x}{\sqrt{x}-\sqrt{a-x}}=\frac{b}{\sqrt{x}}$, to find the roots. *Roots*, $x=\frac{b\pm\sqrt{b^2-2ab}}{2}$.

Suggestion.—Add the fractions in the first member, and $\frac{2x\sqrt{x}}{2x-a}=\frac{b}{\sqrt{x}}$. Whence $2x^2=2bx-ab$.

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106. Cor. 3.—*Upon the principle that the middle term of a trinomial square is twice the product of the square roots of the other two* (**94, 95,** PART I.), *we can often complete the square more advantageously than by the regular rule.*

46. Solve $4x^2+16x=33$.

Solution.—Dividing $16x$ by twice the square root of $4x^2$, *i. e.*, by $4x$, and adding the square of the quotient, $(4)^2$, to each member, $4x^2+16x+16=49$. Extracting the root, $2x+4=\pm 7$. $\therefore x=\frac{3}{2}$, and $-\frac{11}{2}$.

47. Solve $8x^2-12x=36$.

Suggestion.—Divide the equation by 2, and proceed as above. $4x^2-6x+(\frac{3}{2})^2=\frac{81}{4}$, $2x-\frac{3}{2}=\pm\frac{9}{2}$, and $x=3$, and $-1\frac{1}{2}$. In this example the regular method is better.

48. Solve $3x^2+2x=5$.

Suggestion.—Multiply by 3 and $9x^2+6x=15$. Whence $9x^2+6x+1=16$. $x=1$, and $-\frac{5}{3}$.

49. Solve $110x^2-21x = -1$. *Result,* $x = \frac{1}{10}$, and $\frac{1}{11}$.

Suggestion.—Multiply by 110, and $(110)^2x^2-21\times 110x = -110$. Whence $110^2x^2-21\times 110x+(\frac{21}{2})^2 = \frac{1}{4}$, and $110x-\frac{21}{2} = \pm\frac{1}{2}$.

50. Solve $3x^2+5x = 2$.

Suggestion. $(3)^2x^2+3\cdot 5x+(\frac{5}{2})^2 = 6+\frac{25}{4} = \frac{49}{4}$. Whence $3x+\frac{5}{2} = \pm\frac{7}{2}$. $\therefore x = \frac{1}{3}$, and -2.

Scholium.—It appears that by this method the term to be added to complete the square is the square of $\frac{1}{2}$ the coefficient of the first power of x. Therefore when this coefficient is odd, fractions arise. These can always be avoided by doubling the equation when this coefficient is odd, before completing the square.

51. Solve $3x^2-7x = 40$.

Solution.—Multiplying by 2 to avoid fractions, $6x^2-14x = 80$. $(6)^2x^2-6\cdot 14x+(7)^2 = 49+480 = 529$. $6x-7 = \pm 23$. $x = 5$, and $-2\frac{2}{3}$.

52. Solve $3x(x+5) = 96+4x(1-x)$, and verify.

Suggestion.—Perform as few multiplications as possible. When the square is completed, this stands $(14)^2x^2+14\cdot 22x+(11)^2=14\cdot 192+121 = 2809$.

53. Solve $\frac{5}{7}x^2+\frac{7}{5}x+\frac{73}{140} = 0$.

54. Solve $\dfrac{1}{x-2} - \dfrac{2}{x+2} = \dfrac{3}{5}$.

55. Solve $(ax-b)(bx-a) = c^2$.

Suggestion. $abx^2-(a^2+b^2)x = c^2-ab$. $2abx^2-(m) = 2c^2-2ab$, letting (m) stand for the term which becomes the middle term of the trinomial square and which disappears in the subsequent process. Then $(2ab)^2x^2 - (m) + (a^2+b^2)^2 = (a^2+b^2)^2 + 4abc^2 - 4a^2b^2$. $\therefore x = \dfrac{a^2+b^2 \pm \sqrt{(a^2-b^2)^2+4abc^2}}{2ab}$.

56. Solve $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}} = \dfrac{x}{a-x}$. *Roots,* $x = a$, and $\frac{1}{2}a$.

Suggestion.—Clear of fractions and condense.

SECTION III.

EQUATIONS OF OTHER DEGREES WHICH MAY BE SOLVED AS QUADRATICS.

***107.* Prop. 1.**—*Any Pure Equation* (i. e., *one containing the unknown quantity affected with but one exponent*) *can be solved in a manner similar to a Pure Quadratic.*

Demonstration.—In any such equation we can find the value of the unknown quantity affected by its exponent, as if it were a simple equation. If then the unknown quantity is affected with a positive integral exponent it can be freed of it by evolution; if its exponent be a positive fraction it can be freed of it by extracting the root indicated by the numerator of the exponent, and involving this root to the power indicated by the denominator. If the exponent of the unknown quantity is negative it can be rendered positive by multiplying the equation by it with a numerically equal positive exponent. Q. E. D.

EXAMPLES.

1. Solve $y - \frac{7}{y^2} = \frac{1}{8}y$.

Suggestion. $y^3 = 8$. $\therefore y = 2$. Why not put the ± sign before the 2?

2. Solve $\frac{6}{y^3+8} = -\frac{3}{60+\frac{1}{2}y^3}$. *Result,* $y = -4$.

3. Solve $3x^{\frac{1}{3}} - 5 = 2x^{\frac{1}{3}}$. *Result,* $x = 125$.

4. Solve $\frac{x^{\frac{3}{5}}}{2} + 1 = 5\left(\frac{x^{\frac{3}{5}}}{4} - 1\right)$. *Result,* $x = 32$.

5. Solve $12x^{-\frac{2}{3}} + \frac{5}{3} = \frac{2}{x^{\frac{2}{3}}} + \frac{25}{9}$. *Result,* $x = 27$.

6. Given $\left\{x+(4ax+17a^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = x^{\frac{1}{2}}+a^{\frac{1}{2}}$, to find the value of x. *Result*, $x = 16a$.

7. Solve $3ax^{\frac{m}{n}} = 2ax^{\frac{m}{n}} + 5b$. *Result*, $x = \sqrt[m]{\left(\frac{5b}{a}\right)^n}$.

8. Solve $a-1 = bx^{\frac{m}{n}-1} - x^{\frac{m}{n}-1}$.

Result, $x = \left(\frac{a-1}{b-1}\right)^{\frac{n}{m-n}}$.

9. Solve $x^{\frac{3}{4}} = 27$. Also $x^{\frac{2}{5}} = 4$. Also $y^{\frac{5}{3}} = 32$.
Roots, 81, ± 32, and 8.

Query.—Why the ± sign in one case and not in the others?

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108. Prop. 2.—*Any equation containing one unknown quantity affected with only two different exponents, one of which is twice the other, can be solved as an Affected Quadratic.*

Demonstration.—Let m represent any number, positive or negative, integral or fractional; then the two exponents will be represented by m and $2m$; and the equation can be reduced to the form $x^{2m}+px^m = q$. Now let $y=x^m$, whence $y^2=x^{2m}$, whatever m may be. Substituting, we have $y^2+py = q$, whence $y = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4}+q}$. But $y = x^m$; hence $x = \left(-\frac{p}{2} \pm \sqrt{\frac{p^2}{4}+q}\right)^{\frac{1}{m}}$. Q. E. D.

EXAMPLES.

Ex. 1. Solve $3x^3+42x^{\frac{3}{2}} = 3321$.

Solution.—Let $y = x^{\frac{3}{2}}$, whence $y^2 = x^3$; and $3y^2+42y = 3321$. From this $y = 27$, and -41. Taking the first value, $x^{\frac{3}{2}} = 27$. $\therefore x = 9$. Taking the second $x^{\frac{3}{2}} = -41$. $\therefore x = \sqrt[3]{1681}$. We therefore find that $x = 9$, and $\sqrt[3]{1681}$.

2. Solve $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$. $x=\pm 8$, and $\pm(-11)^{\frac{3}{2}}$.

Query.—How many values? Which are imaginary?

3. Solve $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39$. $x=729$, and $\left(\frac{13}{4}\right)^6$.

4. Solve $3x^6+42x^3=3321$. $x=3$, and $-\sqrt[3]{41}$.

5. Solve $\frac{8}{x^3}+2=\frac{17}{x^{\frac{3}{2}}}$. $x=4$, and $\frac{1}{2}\sqrt[3]{2}$.

Scholium.—It is not necessary to substitute another letter for the unknown quantity as given in such examples. Thus, in *Ex.* 3, doubling, to avoid fractions, $8x^{\frac{1}{3}}+2x^{\frac{1}{6}}=78$. Completing the square $8^2x^{\frac{1}{3}}+(m)+1=8\cdot 78+1=625$. Extracting root, $8x^{\frac{1}{6}}+1=\pm 25$, $x^{\frac{1}{6}}=3$, and $-\frac{13}{4}$. $\therefore x=729$, and $\left(\frac{13}{4}\right)^6$.

[**Note.**—Solve the next six without substituting.]

6. Solve $x^{10}+31x^5=32$. $x=1$, and -2.

7. Solve $x^{\frac{1}{n}}+13x^{\frac{1}{2n}}=14$. $x=1$, and $(-14)^{2n}$.

8. Solve $3x^{\frac{3}{2}}-4x^{\frac{3}{4}}=7$. $x=\left(\frac{7}{3}\right)^{\frac{4}{3}}$, and 1.

9. Solve $3x+2\sqrt{x}=1$.

10. Solve $\sqrt{2x}-7x=-52$. $x=8$, and $\frac{338}{49}$.

11. Solve $x+2\sqrt{ax}+c=0$.

$x=\left\{-\sqrt{a}\pm\sqrt{a-c}\right\}^2$.

12. Solve $x\sqrt{\frac{6}{x}-x}=\frac{1+x^2}{\sqrt{x}}$.

$x=\pm\sqrt{1\pm\frac{1}{2}\sqrt{2}}$.

Suggestion.—The first member may be written $\frac{\sqrt{6x^2-x^4}}{\sqrt{x}}$. Hence dropping the denominator, $\sqrt{x}$, and squaring, $6x^2-x^4=1+2x^2+x^4$.

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109. Prop. 3.—*Equations may frequently be put in the form of a quadratic by a judicious grouping of terms containing the unknown quantity, so that one group shall be the square root of the other.*

Demonstration.—This proposition will be established by a few examples, as it is not a general truth, but only points out a special method.

EXAMPLES.

Ex. 1. Solve $2x^2+3x-5\sqrt{2x^2+3x+9}+3=0$.

Solution.—Add 6 to each member and arrange thus, $(2x^2+3x+9)-5(2x^2+3x+9)^{\frac{1}{2}}=6$. Put $(2x^2+3x+9)^{\frac{1}{2}}=y$, and the equation becomes $y^2-5y=6$. Whence $y=6$, and -1. Taking $y=6$, $2x^2+3x+9=36$. Whence $x=3$, and $-4\frac{1}{2}$. Taking $y=-1$, $2x^2+3x+9=1$. Whence $x=\frac{-3\pm\sqrt{-55}}{4}$.

2. Given $(2x+6)^{\frac{1}{2}}+(2x+6)^{\frac{1}{4}}=6$, to find the values of x.

Suggestion.—Put $y=(2x+6)^{\frac{1}{4}}$; whence $y^2+y=6$, $y=2$, and -3. $\therefore$ $2x+6=16$, and also $2x+6=81$. $x=5$, and $37\frac{1}{2}$.

Query.—Will the value $x=37\frac{1}{2}$ verify? Why?

Ans.—Since $(2x+6)^{\frac{1}{2}}$ and $(2x+6)^{\frac{1}{4}}$ are *even* roots, their signs are strictly ambiguous, though not so expressed in the example. Substituting for x, $37\frac{1}{2}$, the equation becomes $(81)^{\frac{1}{2}}+(81)^{\frac{1}{4}}=6$. If now we regard $(81)^{\frac{1}{4}}=-3$, as it is, as really as it is $+3$, the value verifies. Such cases are frequent.

3. Given $(x+12)^{\frac{1}{2}}=6-(x+12)^{\frac{1}{4}}$, to find the values of x, and verify both values.

4. Given $\frac{1}{(2x-4)^2}=\frac{1}{8}+\frac{2}{(2x-4)^4}$, to find the values of x.

Values, $x=3$, and 1.

Suggestion.—Put $(2x-4)^2=y$.

5. Solve $x+5-\sqrt{x+5}=6$. $x=4$, and -1.

6. Solve $2\sqrt{x^2-3x+11}=x^2-3x+8$.

Suggestion. $x^2-3x+8-2\sqrt{x^2-3x+11}=0$. Add 3 to each member and $x^2-3x+11-2\sqrt{x^2-3x+11}=3$. Put $\sqrt{x^2-3x+11}=y$, and $y^2-2y=3$. $\therefore$ $x=2$, 1, and $\frac{1}{2}(3\pm\sqrt{-31}.)$

7. Solve $(x^2-9)^2=3+11(x^2-2)$.

$x=\pm 5$, and ± 2.

8. Solve $\left(x+\frac{8}{x}\right)^2+x=42-\frac{8}{x}$.

$x=4$, 2, and $\frac{1}{2}(-7\pm\sqrt{17})$.

9. Solve $x^4\left(1+\frac{1}{3x}\right)^2-(3x^2+x)=70$.

Suggestion. $x^4(1+\frac{1}{3x})^2=\frac{1}{9}(3x^2+x)^2$. Hence the equation may be written $(3x^2+x)^2-9(3x^2+x)=630$.

Results, $x=3$, $-3\frac{1}{3}$, and $\frac{1}{6}(-1\pm\sqrt{-251})$.

10. Solve $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$. *Roots*, $x=4$, and 1.

Suggestion.—Divide by $x+\sqrt{x}$, and $\frac{1}{x-\sqrt{x}}=\frac{x-\sqrt{x}}{4}$. Hence $4=(x-\sqrt{x})^2$, or $x-\sqrt{x}=\pm 2$.

***110.* Cor.**—THE FORM OF THE COMPOUND TERM *may sometimes be found by transposing all the terms to the first member, arranging them with reference to the unknown quantity, and extracting the square root.*

In trying this expedient, if the highest exponent is not even it must be made so by multiplying the equation by the unknown quantity. In like manner the coefficient of this term is to be made a perfect square. When the process of extracting the root terminates, if the root found is a part, or a factor, or a factor of a part of the remainder, the root may be made the compound term.

11. Solve $\frac{x}{14} - \frac{30}{7x^3} + \frac{12+\frac{1}{2}x}{3x} = \frac{7}{2x^2} + 1\frac{1}{6}$.

Solution.—Cleared of fractions, transposed and arranged, this becomes $3x^4 - 42x^3 + 168x^2 - 147x - 180 = 0$. Dividing by 3, $x^4 - 14x^3 + 56x^2 - 49x - 60 = 0$.

Extracting square root,

$$\begin{array}{r|l|l}
 & x^4 - 14x^3 + 56x^2 - 49x - 60 & \underline{x^2 - 7x.} \\
 & \underline{x^4} & \\
\underline{2x^2 - 7x} & -14x^3 + 56x^2 & \\
 & \underline{-14x^3 + 49x^2} & \\
 & \underline{7x^2 - 49x - 60} & \\
 & 7(x^2 - 7x) - 60 &
\end{array}$$

After obtaining two terms of the root we observe the root itself as a factor in part of the remainder. The equation may therefore be written $(x^2 - 7x)^2 + 7(x^2 - 7x) - 60 = 0$. This is solved as before. $x = 4, 3$, and $\frac{1}{2}(7 \pm \sqrt{69})$.

12. Solve $x^4 - 12x^3 + 44x^2 - 48x = 9009$.

One value of x is 13.

13. Solve $x^3 - 6x^2 + 11x - 6 = 0$, $x = 1, 2, 3$.

14. Solve $4x^4 + \frac{x}{2} = 4x^3 + 33$.

$x = 2, -\frac{3}{2}$, and $\frac{1}{4}(1 \pm \sqrt{-43})$.

15. Solve $x^4 - 2x^3 + x = a$. $x = \frac{1}{2} \pm \sqrt{\frac{3}{4} \pm \sqrt{a + \frac{1}{4}}}$.

16. Solve $x^4 + x^3 - 4x^2 + x + 1 = 0$.

$x = 1$, and $\frac{-3 \pm \sqrt{5}}{2}$.

Solution.--Dividing by x^2, $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0$, which may be written $x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6$, or $\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6$. Whence the compound term appears.

111. Prop. 4.—*When an equation is reduced to the form* $x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}\;-\;-\;-\;+L=0$, *the roots, with their signs changed, are factors of the absolute (known) term,* L.

Demonstration.—1st. The equation being in this form, if a is a root, the equation is divisible by $x-a$. For, suppose upon trial $x-a$ goes into the polynomial $x^n+Ax^{n-1}+$, etc., Q times *with a remainder* R. (Q represents any series of terms which may arise from such a division, and R, any remainder.) Now, since the quotient multiplied by the divisor, + the remainder equals the dividend, we have $(x-a)\,Q+R=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\;-\;-\;-\;+L$. But this polynomial $=0$. Hence $(x-a)Q+R=0$. Now, by hypothesis a is a root, and consequently $x-a=0$. Whence $R=0$, or there is no remainder.

2nd. If now $x-a$ exactly divides $x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}\text{-}\text{-}+L$, a must exactly divide L, as readily appears from considering the process of division. Hence $-a$ is a factor of L, a being a root of the equation. Q. E. D.

EXAMPLES.

Ex. 1. Solve $2x^4-68x=32-17x^3$.

Solution.—This may be written $x^4+\frac{17x^3}{2}-34x-16=0$. By Prop. 4 the roots of this equation are factors of 16. We therefore try in order ±1, ±2, ±4, ±8, ±16 (all the integral factors of 16) till we find whether there is an integral root. We see at once that neither +1 nor −1 satisfies the equation. Trying +2 we find it is a root. Hence $x-2$ is the divisor sought. Dividing $x^4+\frac{17x^3}{2}-34x-16=0$, by $x-2$, we have $x^3+\frac{21}{2}x^2+21x+8=0$. But as -2 satisfies the equation as well as $+2$, $x+2$ is a divisor. Dividing

$x^3+\frac{21}{2}x^2+21x+8=0$ by $x+2$, we have $x^2+\frac{17}{2}x+4=0$. From this equation $x=-8$, and $-\frac{1}{2}$. The roots therefore are 2, -2, -8, $-\frac{1}{2}$.

2. Solve $x-1=2+\frac{2}{\sqrt{x}}$.

Suggestion.—Put $y=\sqrt{x}$, and clearing of fractions and transposing, $y^3-3y-2=0$. If there are integral roots of this equation they are ±1, or ±2. $+1$ does not satisfy the equation and hence is not a root. But -1 does. Hence $y+1$ is a divisor, Dividing, we get $y^2-y-2=0$. Whence $y=2$, and -1. $\therefore$ The roots of the equation $y^3-3y-2=0$ are -1, 2, and -1, there being two *equal* roots. Now as $y=\sqrt{x}$, $x=1$, and 4.

3. Solve $x^3-3x^2+x+2=0$.

The irrational roots are $\frac{1}{2}(1\pm\sqrt{5})$.

4. Solve $x^3=6x+9$.

Imaginary roots, $\frac{1}{2}(-3\pm\sqrt{-3})$.

5. Solve $x^3-6x^2-x+30=0$.

6. Solve $x^3-17x^2+59x+77=0$.

7. Solve $x+7x^{\frac{1}{2}}=22$. (Put $x^{\frac{1}{2}}=y$.)

Imaginary roots, $29\mp7\sqrt{-10}$.

8. Solve $x^4-2x^3-13x^2-4x-30=0$.

9. Solve $x=\frac{12+8\sqrt{x}}{x-5}$.

Imaginary roots, $\frac{1}{2}(-3\mp\sqrt{-7})$.

10. Solve $x^3-2x^2+4x+7=0$.

11. Solve $x^2+y=7$, and $y^2+x=11$.

Suggestions.—From the 1st, $x^2=7-y$, and from the 2d, $x^2=121-22y^2+y^4$. Whence $y^4-22y^2+y+114=0$. From this we readily find $y=3$; whence from either of the given equations we can find a value of x.

There are, however, three other values of each of the unknown quantities. These can not be found without some knowledge of Higher Equations.

This example is inserted here because it is so often propounded to teachers. It is not a proper example to propose for solution by the lower algebra, *i. e.*, by the method of quadratics. (See **113.**)

12. Solve $x^4+x^2-2=0$.
Also $x^3-2x^2+x-2=0$.
Also $x^3+6x^2+11x+6=0$.

13. Solve $\dfrac{18}{x^2}+\dfrac{81-x^2}{9x}=\dfrac{x^2-65}{72}$.

Suggestion.—Putting this in the form required, the absolute term is -1296; the integral factors of which are very numerous. But trying ±1, ±2, ±3, ±4, ±6, ±8, ±9, we find -4 and 9 to be two of the roots.

The method of solving this by putting it in form so as to complete the square of both members, is to multiply by 2, and add $\dfrac{8x}{36}+\dfrac{81}{36}$ to both members, obtaining $\dfrac{36}{x^2}+\dfrac{18}{x}+\dfrac{9}{4}=\dfrac{x^2}{36}+\dfrac{2x}{9}+\dfrac{4}{9}$. Extracting root, $\dfrac{6}{x}+\dfrac{3}{2}=\pm\left(\dfrac{x}{6}+\dfrac{2}{3}\right)$. Whence $x=9, -4, -4$, and -9, there being two roots -4.

14. Given $x-3=\dfrac{3+4\sqrt{x}}{x}$ to find the roots.

Suggestion.—The roots being all surds and imaginaries in this equation cannot be found by the principle in the proposition. The following special expedient will, however, effect a solution:

Clear of fractions and add $x+1$ to each member and $x^2-2x+1=4+4\sqrt{x}+x$.

14. Solve the following by the principle in the proposition:

$$x+1=\frac{x^2-1}{2x}+\frac{3}{x^2}(4x+10).$$

$$\frac{x-1}{x+5}=\frac{3}{x}+\frac{15}{x}\cdot\frac{x+1}{x+5}.$$

$$x^4+x^3-4x^2+x+1=0. \quad \text{(See } \textit{Ex.}\ 16,\ \textit{Prop.}\ 3.\text{)}$$

[**Note.**—Many of the examples under PROP. 3 can be readily solved in this manner.]

SECTION IV.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE BETWEEN TWO UNKNOWN QUANTITIES.

***112.* Prop. 1.**—*Two equations between two unknown quantities, one of the second degree and the other of the first, may always be solved as a quadratic.*

Demonstration.—The general form of a *Quadratic Equation* between two unknown quantities is

$$ax^2+bxy+cy^2+dx+ey+f=0,$$

since in every such equation all the terms in x^2 can be collected into one, and its coefficient represented by a; all those in xy can also be collected into one, and its coefficient represented by b, etc.

The general form of an equation of the *First Degree* between two unknown quantities is

$$a'x+b'y+c'=0.$$

Now, from the latter $x=\frac{-b'y-c'}{a'}$, which substituted in the former gives no term containing a higher power of y than the second, and hence the resulting equation is a quadratic. Q. E. D.

EXAMPLES.

Ex. 1. Given $x-\dfrac{x-y}{2}=4$, and $y-\dfrac{x+3y}{x+2}=1$.

Suggestion.—From the first $x=8-y$. Substituting this value of x in the second, we have $y-\dfrac{8-y+3y}{8-y+2}=1$, or $y-\dfrac{8+2y}{10-y}=1$; whence $y^2-9y=-18$, and $y=6$ and 3.

2. Given $x+y=7$, and $x^2+2y^2=34$. Verify.

3. Given $\dfrac{1}{x}+\dfrac{1}{y}=2$, and $x+y=2$. Verify.

4. Given $x+y=100$, and $xy=2400$.

5. Given $2x+3y=37$, and $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{14}{45}$.

6. Given $2x^2+xy-5y^2=20$, and $2x-3y=1$.

7. Given $x+\frac{1}{3}=\dfrac{2x+y}{3}$, and $\dfrac{x+y}{x}=\dfrac{4x-y}{2}$.

8. Given $.1y+.125x=y-x$, and $y-.5x=.75xy-3x$.

Results, $x=0$, and 4; and $y=0$, and 5.

9. Given $.3x+.125y=3x-y$, and $3x-.5y=2.25xy+3y$.

Results, $x=0$, and -1; and $y=0$, and $-2\frac{2}{5}$.

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113. Prop. 2.—*In general, the solution of two quadratics between two unknown quantities, requires the solution of a biquadratic.*

Demonstration.—Two General Equations between two unknown quantities have the forms

$$(1)\qquad ax^2+bxy+cy^2+dx+ey+f=0, \text{ and}$$
$$(2)\qquad a'x^2+b'xy+c'y^2+d'x+e'y+f'=0.$$

From (1), $x=-\dfrac{by+d}{2a}\pm\sqrt{\dfrac{(by+d)^2}{4a^2}-\dfrac{cy^2+ey+f}{a}}$.

Now, to substitute this value of x in equation (2), it must be squared, and also, in another term, multiplied by y, either of which operations produce rational terms containing y^2, and a radical of the second degree. Then, to free the resulting equation of radicals will require the squaring of terms containing y^2, which will give terms in y^4, as well as other terms. Q. E. D.

[**Note.**—Since it is not the purpose of this treatise to embrace the resolution of the higher equations, only such special cases of Simultaneous Quadratics with two unknown quantities, will be introduced, as can be resolved by the methods of quadratics.]

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114. **Prop. 3.**—*Two Homogeneous Quadratic Equations between two unknown quantities can always be solved by the method of quadratics, by substituting for one of the unknown quantities the product of a new unknown quantity into the other.*

Definition.—A Homogeneous Equation is one in which each term contains the same number of factors of the unknown quantities. $2x^2 - 3xy - y^2 = 16$ is homogeneous. $3x^2 - 2y + y^2 = 10$ is not homogeneous.

Demonstration.—The truth of this proposition will be more readily apprehended by means of a particular example. Taking the two homogeneous equations $x^2 - xy + y^2 = 21$, and $y^2 - 2xy + 15 = 0$. Let $x = vy$, v being a new unknown quantity, called an auxiliary, whose value is to be determined. Substituting in the given equations, we have $v^2y^2 - vy^2 + y^2 = 21$, and $y^2 - 2vy^2 = -15$. From these we find $y^2 = \frac{21}{v^2 - v + 1}$, and $y^2 = \frac{15}{2v - 1}$. Equating these values of y^2, $\frac{21}{v^2 - v + 1} = \frac{15}{2v - 1}$; whence $42v - 21 = 15v^2 - 15v + 15$. This latter equation is an affected quadratic, which solved for v, gives $v = 3$, and $\frac{4}{5}$. Knowing the values of v we readily determine those of y from $y^2 = \frac{15}{2v - 1}$, and find $y = \pm\sqrt{3}$ when $v = 3$, and $y = \pm 5$ when $v = \frac{4}{5}$. Finally as $x = vy$, its values are $x = \pm 3\sqrt{3}$, and ± 4.

By observing the substitution of vy for x in this solution it is seen that it brings the square of y in every term containing the unknown quantities, in each equation, and hence enables us to find two values of y^2 in terms of v. It is easy to see that this will be the case in any homogeneous quadratic with two unknown quantities, for we have in fact, in the first of the given equations, all the variety of terms which such an equation can contain. Again, that the equation in v will not be higher than the second degree is evident, since the values of y^2 consist of known quantities for numerators, and can have denominators of only the second, or second and first degrees with reference to v. Whence v can always be determined by the method of quadratics; and being determined, the value of y is obtained from a *pure* quadratic ($y^2 = \frac{15}{2v-1}$, in this case), and that of x from a *simple* equation ($x = vy$ in this case).

EXAMPLES.

Ex. 1. Given $3x^2+xy = 18$, and $4y^2+3xy = 54$.
Roots, $x = \pm 2$, and $\pm 2\sqrt{3}$; and $y = \pm 3$, and $\pm 3\sqrt{3}$.

2. Given $x^2+xy+2y^2 = 74$, and $2x^2+2xy+y^2 = 73$.
Roots, $x = \pm 3$, and ∓ 8; and $y = \pm 5$.

3. Given $x^2+3xy = 54$, and $xy+4y^2 = 115$.
Roots, $x = \pm 3$, and ± 36; and $y = \pm 5$, and $\pm 11\frac{1}{2}$.

4. Given $2x^2+3xy = 26$, and $3y^2+2xy = 39$.
Roots, $x = \pm 2$, and $y = \pm 3$. The other roots are ∞.

5. Given $x^2-4y^2 = 9$, and $xy+2y^2 = 3$.
Roots, $x = \pm\frac{15}{\sqrt{21}}$; and $y = \pm\frac{3}{\sqrt{21}}$. The other roots are ∞.

6. Given $3x^2+xy-9 = 9$, and $4y^2+3xy-4 = 50$.
(See *Ex.* 1.) *Roots*, $x = \pm 2$, $y = \pm 3$.

7. Given $x^2-xy = 70$, and $xy-y^2 = 12$.

8. Given $x^2+xy = 84$, and $x^2-y^2 = 24$.

***115.* Prop. 4.**—*When the unknown quantities are similarly involved in two quadratic, or even higher equations, the solution can often be effected as a quadratic, by substituting for one of the unknown quantities the sum of two others, and for the other unknown quantity the difference of these new quantities.*

[**Note.**—As this and the following are merely *special expedients*, they need no demonstrations other than is furnished by applying them to examples.]

EXAMPLES.

Ex. 1. Given $x^2+y^2=52$, and $x+y+xy=34$.

Solution.—In these equations x and y are similarly involved, and hence I try the expedient of putting $x=m+n$, and $y=m-n$, whence $x^2+y^2=2m^2+2n^2=52$, and $x+y+xy=2m+m^2-n^2=34$. Now, from the two equations $m^2+n^2=26$, and $\underline{2m+m^2-n^2=34}$, by adding I have $2m+2m^2=60$, whence I find $m=5$, and -6. Substituting these values in $m^2+n^2=26$, $n=\pm 1$, and $\pm\sqrt{-10}$. Whence the *real* values of x are found to be 6, and 4; and of y, 4, and 6.

2. Given $x^2+x+y=18-y^2$, and $xy=6$.

Rational roots, $x=3$, and 2; and $y=2$, and 3.

3. Given $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3}$, and $x^2+y^2=45$.

Suggestion.—Using the same notation as above, $\dfrac{m}{n}+\dfrac{n}{m}=\dfrac{10}{3}$, and $2m^2+2n^2=45$; whence $3m^2+3n^2=10mn$, and we have $4mn=27$, or $m=\dfrac{27}{4n}$. $m=\pm\dfrac{9}{2}$, and $\pm\dfrac{3}{2}$; $n=\pm\dfrac{3}{2}$, and $\pm\dfrac{9}{2}$. The roots are $x=\pm 6$: and $y=\pm 3$.

4. Given $4(x+y)=3xy$, and $x+y+x^2+y^2=26$.

Roots, $x=4$, and 2; $y=2$, and 4. Also $x=-\frac{13}{6}\pm\frac{1}{6}\sqrt{377}$, and $y=-\frac{13}{6}\mp\frac{1}{6}\sqrt{377}$.

Definition and Scholium.—It will be observed that the above equations are of the second degree, and that they have the unknown quantities similarly involved; that is, in the last, for example, in the first member there is $+4x$, and also $+4y$; in the second member x and y are multiplied together; in the second equation there is $+x$, also $+y$; there is $+x^2$, and also $+y^2$. Such equations can usually be readily solved in this manner. But the equations $x^2+y^2 = \frac{5}{2}xy$, and $x-y=\frac{1}{4}xy$ have the unknown quantities similarly involved in the first but *dissimilarly* in the second. There is $+x$ in the second, but no $+y$, hence they are not similarly involved. Whether the solution of such equations will be facilitated by this expedient can be ascertained only by trial. In this case the expedient will be found successful.

5. Given $x^2+y^2=\frac{5}{2}xy$, and $x-y=\frac{1}{4}xy$.

Roots, $x=0$, 4, and -2; $y=0$, 2, and -4.

6. Given $x^2+xy+4y^2=6$, and $3x^2+8y^2=14$.

Roots, $x=\pm 2$, and $\mp\frac{1}{5}\sqrt{10}$; and $y=\pm\frac{1}{2}$, and $\pm\frac{2}{5}\sqrt{10}$.

7. Given $x^2-4y^2=9$, and $xy+2y^2=3$.

Suggestion.—The student will find by experiment that the above expedient is of no service in this example. The example is readily solved by finding the value of x from the first equation and substituting it in the second, thus obtaining $y\sqrt{4y^2+9}=3-2y^2$, or $4y^4+9y^2=9-12y^2+4y^4$. Whence $21y^2=9$, and $y=\pm\sqrt{\frac{3}{7}}$. Or the equations can be treated as in (**114**), they being homogeneous.

8. Given $\frac{x^2}{y}+\frac{y^2}{x}=18$, and $x+y=12$.

Suggestion.—In these equations the unknown quantities are similarly involved, and although the first is of the third degree the expedient of the proposition is successful. $x=8$, and 4; and $y=4$, and 8.

Scholium.—In all symmetrical equations the value of the unknown quantities must, of course, be the same numerically, but taken in the reverse order, since the letters can change places in the equation without altering the equations. When, therefore, in such equations the values of one of the unknown quantities are found the values of the other are known.

9. Given $x^2-x^2y^2+y^2=19$, and $x-xy+y=4$.

Suggestion.—Putting $x=m+n$, and $y=m-n$, $x^2+y^2=2m^2+2n^2$. and $x^2y^2=(m+n)^2(m-n)^2=(m+n)(m-n)\times(m+n)(m-n)=(m^2-n^2)^2$. Hence $2m^2+2n^2-(m^2-n^2)^2=19$. From the second equation, $n^2=4+m^2-2m$.

Roots, $x=\frac{1}{4}(9\pm\sqrt{73})$; and $y=\frac{1}{4}(9\mp\sqrt{73})$.

10. Given $x+y=11$, and $x^3+y^3=407$.

Roots, $x=7$, and 4; and $y=4$, and 7,

11. Given $x-y=3$, and $x^4+y^4=641$.

Roots, $x=5$, and -2; $y=2$, and -5.

SPECIAL EXPEDIENTS.

116. Many equations of other degrees than the second, and which do not fall under the preceding cases, may still be solved as quadratics by means of special artifices. For these artifices the student must depend upon his own ingenuity, after having studied some examples as specimens. These methods are so restricted and special that it is not expedient to classify them; in fact, every expert algebraist is constantly developing new ones.

Ex. 1. Given $x^{\frac{1}{4}}+y^{\frac{1}{3}}=5$, and $x^{\frac{1}{2}}+y^{\frac{2}{3}}=13$.

Suggestion.—In case of fractional exponents, it will usually be found expedient for the learner to put the unknown quantities with the lowest exponents, equal to the first powers of new unknown quantities, and thus make the exponents integral. Thus, putting $x^{\frac{1}{4}}=m$, and $y^{\frac{1}{3}}=n$, we have $x^{\frac{1}{2}}=m^2$, and $y^{\frac{2}{3}}=n^2$, Whence the equations become $m+n=5$, and $m^2+n^2=13$. These equations are readily solved by methods already learned, and we find $m=3$, and

2, and $n = 2$, and 3. Hence $x^{\frac{1}{4}} = 3$, gives $x = 81$; and $x^{\frac{1}{4}} = 2$, gives $x = 16$. Also $y^{\frac{1}{3}} = 2$, gives $y = 8$; and $y^{\frac{1}{3}} = 3$, gives $y = 27$.

2. Given $x^{\frac{1}{4}} + y^{\frac{1}{5}} = 6$, and $x^{\frac{3}{4}} + y^{\frac{3}{5}} = 126$.
Roots, $x = 625$, and 1, and $y = 1$, and 3125.

3. Given $x^{\frac{2}{3}} y^{\frac{3}{2}} = 2y^2$, and $8x^{\frac{1}{3}} - y^{\frac{1}{2}} = 14$.

4. Given $x - y = \sqrt{x} + \sqrt{y}$, and $x^{\frac{3}{2}} - y^{\frac{3}{2}} = 37$.

Suggestion.—Observe that both members of the first are divisible by $\sqrt{x} + \sqrt{y}$, giving $\sqrt{x} - \sqrt{y} = 1$. $x = 16$, and 9; $y = 9$, and 16.

5. Given $x^2 + x + y = 18 - y^2$, and $xy = 6$.

Suggestion.—From the first, by adding twice the second, we may write $x^2 + 2xy + y^2 + x + y = 30$, or $(x+y)^2 + (x+y) = 30$. $\therefore\ x + y = 5$, and -6.

6. Given $x^2 + y^2 = 52$, and $x + y + xy = 34$.
Roots, $x = 6$, 4, and $-6 \pm \sqrt{-10}$; $y = 4$, 6, and $-6 \mp \sqrt{-10}$.

7. Given $x^2 + y^2 + 4\sqrt{x^2 + y^2} = 45$, and $x^4 + y^4 = 337$.

Suggestion.—In the first, put $\sqrt{x^2 + y^2} = v$, whence $v^2 + 4v = 45$; and $v = 5$, and -9. $x = 3$, and 4; $y = 4$, and 3.

8. Given $\dfrac{x^4}{y^2} + 2\dfrac{x^2}{y} = 9\frac{39}{49}$, and $x^2 + y^2 = 65$.

Suggestion.—In the first, put $\dfrac{x^2}{y} = v$, whence $v^2 + 2v = 9\frac{39}{49}$.

Real and rational roots, $x = 4$, $y = 7$.

9. Given $x^2 + 2xy + y^2 + 2x = 120 - 2y$, and $xy - y^2 = 8$.
Roots, $y = 1$, 4, $-3 - \sqrt{5}$, and $-3 + \sqrt{5}$; $x = 9$, 6, $-9 + \sqrt{5}$, and $-9 - \sqrt{5}$.

10. Given $x^2y^2 = 180 - 8xy$, and $x + 3y = 11$.
Roots, $x = 5$, and 6; $y = 2$, and $\frac{5}{3}$.

11. Given $x^2 + 3x + y = 73 - 2xy$, and $y^2 + 3y + x = 44$.

Suggestion.—Add the two equations together, and proceed as before. $x = 4$, 16, and $-12 \mp \sqrt{58}$; and $y = 5$, -7, and $-1 \pm \sqrt{58}$.

12. Given $xy + xy^2 = 12$, and $x + xy^3 = 18$.

Suggestion.—From the first, $x = \frac{12}{y(1+y)}$; and from the second, $x = \frac{18}{1+y^3}$. $\therefore \frac{12}{y(1+y)} = \frac{18}{1+y^3}$. Dividing denominators by $1 + y$, and numerators by 6, we have $\frac{2}{y} = \frac{3}{1-y+y^2}$; whence $2 - 2y + 2y^2 = 3y$. Hence $x = 2$, and 16; and $y = 2$, and $\frac{1}{2}$.

13. Given $x^2 + xy + y^2 = 26$, and $x^4 + x^2y^2 + y^4 = 364$.

Suggestion.—From the first, $x^2 + y^2 = 26 - xy$; and from the second, by adding x^2y^2 to both members and extracting the square root, $x^2 + y^2 = \sqrt{364 + x^2y^2}$. Equating these values of $x^2 + y^2$, and squaring, we have $676 - 52xy + x^2y^2 = 364 + x^2y^2$, whence $xy = 6$. Squaring this and adding it to the second, and extracting the square root, $x^2 + y^2 = 20$. Also subtracting $3x^2y^2 = 108$ from the second, and extracting the square root, $x^2 - y^2 = 16$. Whence $x^2 = 18$, and $y^2 = 2$.

Another Solution.—Dividing the second by the first, we have $x^2 - xy + y^2 = 14$. Subtracting this result from the first, we have $2xy = 12$. Whence the solution proceeds as above.

[**Note.**—Though this field is illimitable, it is not thought necessary for the learner to pursue special methods farther, inasmuch as what is given will enable him to catch the spirit of such solutions, and no writer in discussing a problem involving processes even as complex as some given above, would fail to give hints at his methods of solution.]

APPLICATIONS.

[**Note.**—One of the most important things to be learned from the following examples is several devices frequently found serviceable in stating a problem, which make the equations arising more simple and easy of solution. These devices are of special necessity in examples involving progressions.]

Ex. 1. What number is that which being divided by the product of its two digits, the quotient is 2, and if 27 is added to it the digits are reversed?

2. There are three numbers, the difference of whose differences is 8; their sum is 41; and the sum of their squares is 699. What are the numbers?

Notation.—Let x be the second number, and y the difference between the second and first, so that $x-y$ represents the first. The first equation is $3x+8=41$, and the second $(11-y)^2+121+(19+y)^2=699$.

3. There are three numbers, the difference of whose differences is 5; their sum is 44; and their product is 1950. What are the numbers?

4. A grocer sold 80 lbs. of mace and 100 lbs. of cloves for $65; but he sold 60 lbs. more of cloves for $20 than he did of mace for $10. What was the price of a pound of each?

5. A and B have each a small field, in the shape of an exact square, and it requires 200 rods of fence to enclose both. The contents of these fields are 1300 square rods. What is the value of each, at $2.25 per square rod?

Ans., One, $900; other, $2,025.

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6. Find two numbers, such that the sum of their squares being subtracted from three times their product, 11 remain; and the difference of their squares being subtracted from twice their product, the remainder is 14. (See **114.**)

7. What two numbers are those whose difference multiplied by the difference of their squares is 32, and whose sum multiplied by the sum of their squares is 272?

8. The difference of two numbers is 2, and the square of their quotient added to four times their quotient is $9\frac{4}{9}$. What are the numbers? *Ans.*, 5 and 3.

9. There are two numbers, whose sum multiplied by the less, is equal to four times the greater, but whose sum multiplied by the greater is equal to 9 times the less. What are the numbers?

10. Find two numbers, such that their product added to their sum shall be 47, and their sum taken from the sum of their squares shall leave 62. *Ans.*, 5 and 7.

11. Find two numbers, such that their sum, their product and the difference of their squares shall be all equal to each other. *Ans.*, $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$, and $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

12. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Ans., $\frac{1}{2}\sqrt{5}$, and $\frac{1}{4}(5+\sqrt{5})$.

13. A person has \$1,300, which he divides into two portions, and loans at different rates of interest, so that the two portions produce equal returns. If the first portion had been loaned at the second rate of interest, it would have produced \$36, and if the second portion had been loaned at the first rate of interest, it would have produced \$49. Required the rates of interest.

Ans., 7 and 6 per cent.

14. The fore wheel of a wagon makes 6 revolutions more than the hind wheel in going 120 yards; but if the periphery of each wheel be increased 1 yard, the fore wheel will make only 4 revolutions more than the hind wheel in going the same distance. What is the circumference of each wheel? *Ans.*, 4 and 5.

15. The sum of two numbers is 8 and the sum of their cubes is 152; what are the numbers?

Suggestion.—Let $x+y$ be one of the numbers, and $x-y$ the other. Then $x=4$, and $2y^3+6xy^2=152$ (**111**).

16. The sum of two numbers is 7, and the sum of their 4th powers is 641. What are the numbers?

17. The sum of two numbers is 6, and the sum of their 5th power is 1056. What are the numbers?

18. The product of two numbers is 24, and their sum multiplied by their difference is 20; find them. *Ans.*, 4 and 6.

19. What two numbers are those whose sum multiplied by the greater is 120, and whose difference multiplied by the less is 16? *Ans.*, 2 and 10.

20. What two numbers are those whose sum added to the sum of their squares is 42, and whose product is 15?
Ans., 3 and 5.

21. A's and B's shares in a speculation altogether amount to $500; they sell out at *par*, A at the end of 2 years, B of 8, and each receives in capital and profits $297. How much did each embark? *Ans.*, A, $275; B, $225.

Suggestion.—Letting x be A's capital, and y B's, A gained $297-x$, and B, $297-y$. And as the gains are proportioned to the products of the respective times into the capitals, $2x : 8y :: 297-x : 297-y$.

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22. What three numbers are those in A. P., whose sum is 120, and the sum of whose squares is 5600? *Ans.*, 20, 40. 60.

Suggestion.—In solving examples involving several quantities in arithmetical progression, it is usually expedient to represent the middle one of the series, when the number of terms is *odd*, by x, and let y be the common difference. If the number of terms is *even*, represent the *two middle terms* by $x-y$, and $x+y$, making the common difference $2y$. Thus the statement of the above problem becomes $x-y+x+x+y=120$, or $3x=120$; and $(x-y)^2+x^2+(x+y)^2=5600$, or $3x^2+2y^2=5600$.

23. What four numbers are those in A. P., the sum of whose squares is 84, and their product 105?

Suggestion.—Call the numbers $x-3y$, $x-y$, $x+y$, and $x+3y$.

24. The sum of five numbers in A. P. is 35, and the sum of their squares 285; find the numbers.

25. What three numbers are those in A. P., the sum of whose squares is 1232, and the square of the mean greater than the product of the two extremes, by 16?

26. Find four numbers in A. P. such that the sum of the squares of the extremes is 4500, and the sum of the squares of the means is 4100.

27. The product of five numbers in A. P. is 945; and their sum is 25. What are the numbers?

Ans., 1, 3, 5, 7, 9.

28. The product of four numbers in A. P. is 280, and the sum of their squares 166; find them.

29. The sum of nine numbers in A. P. is 45, and the sum of their squares 285; find them.

Ans., 1, 2, 3, etc., to 9.

30. The sum of seven numbers in A. P. is 35, and the sum of their cubes 1295; find them. *Ans.*, 2, 3, etc., to 8.

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31. There are three numbers in geometrical progression, whose sum is 52, and the sum of the extremes is to the mean as 10 to 3. What are the numbers?

Ans., 4, 12, and 36.

Suggestion.—Let x be the first term and y the ratio. Then $x+xy+xy^2=52$ and $x+xy^2:xy::10:3$, or $1+y^2:y::10:3$.

32. The sum of three numbers in geometrical progression is 13, and the product of the mean and sum of the extremes is 30. What are the numbers?

Ans., 1, 3, and 9.

Suggestion.—We have $x+xy+xy^2 = 13$, and $(x+xy^2)xy = 30$ From the first $x+xy^2 = 13-xy$, and from the second, $x+xy^2 = \frac{30}{xy}$; whence $13-xy=\frac{30}{xy}$, or $x^2y^2-13xy=-30$.

33. If the seventh and tenth terms of a geometrical progression are 6 and 750 respectively, what are the intermediate terms?

Suggestion.—The equations are $xy^6 = 6$, and $xy^9 = 750$. Divide the second by the first.

34. If the third and fifth terms of a geometrical progression be 75 and 300 respectively, what will the fourth term be?

Ans., 150.

35. If the first and fourth terms of a geometrical progression are 3 and 24 respectively, what are the two intermediate terms?

36. There are four numbers in geometrical progression. The sum of the means is 30, and the product of the extremes 200; what are the numbers?

Suggestion.—Represent the numbers by $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$, in which $\frac{y}{x}$ is the ratio. The equations are $x+y = 30$, and $xy = 200$.

For an odd number of terms, as five, use $\frac{x^3}{y}$, x^2, xy, y^2, $\frac{y^3}{x}$.

37. The sum of three numbers in G. P. is 26, and the sum of their squares is 364; required the numbers.

Suggestion.—The equations are $x^2+xy+y^2 = 26$, and $x^4+x^2y^2+y^4 = 364$, which have already been solved.

SYNOPSIS.

- **QUADRATIC EQUATIONS.**
  - **DEFINITIONS.**
    - **Quadratic Equation.**
      - Pure.—Incomplete.
      - Affected.—Complete.
    - **Root of Equation.**
  - **PURE.**
    - **Prob.** To Solve. *Dem.*
      - *Cor.* 1. Number of Roots. *Dem.*
      - *Cor.* 2. Imaginary Roots.
  - **AFFECTED.**
    - **Prob.** To solve. RULE. *Dem.*
    - **Completing Square.** What?
      - Common method.
      - Special methods.
    - **Cor. 1.** Roots of an Affected Quadratic. *Dem.*
    - **Cor. 2.** To write the root of $x^2+px=q$ directly.
  - **EQUATIONS OF OTHER DEGREES.**
    - **Pure.** *Prop.* 1.
    - **Affected.**
      - *Prop.* 2. *Dem.* When solved as Quad.
      - Expedients.
        - *Prop.* 3. *Cor.*
        - *Prop.* 4. How applied.
  - **SIMULTANEOUS QUADRATICS.**
    - **Prop. 1.** *Dem.*
    - **Prop. 2.** *Dem.*
    - **Prop. 3.** *Dem.*
    - **Prop. 4.** —— Sch.
    - **Expedients.** Enumerate the 7 given.
  - **EXPEDIENTS IN STATING.**
    - **An A. P.**
      - Common method.
      - Special methods.
        - Even number of terms.
        - Odd " " "
    - **A G. P.**
      - Common method.
      - Special methods.
        - Even number of terms.
        - Odd " " "

Test Questions.—If one of two equations between two unknown quantities is of the first degree, and the other of the second, what will be the degree of the resulting equation after eliminating one of the unknown quantities? Prove it. How may such equations be solved? In general, what is the degree of the equation arising from eliminating one unknown quantity from two equations, each of the second degree? Prove it. Mention the several cases given in which such equations can be solved by quadratics, and state the process in each case.

Logarithms

CHAPTER V.

[**Note**.—It is the purpose of this chapter to give a simple, arithmetical view of the nature of logarithms, with some illustrations of their practical utility. For the production of the Logarithmic Series and its use in computing Logarithms, see Appendix II. Enough, however, is here given for practical use in trigonometry, as usually studied, and to enable the student to understand the use of logarithms in ordinary operations.]

117. **A Logarithm** is the exponent by which a fixed number is to be affected in order to produce any required number. The fixed number is called the *Base* of the System.

Illustration.—Let the *Base* be 3 : then the logarithm of 9 is 2 ; of 27, 3 ; of 81, 4 ; of 19683, 9 ; for $3^2=9$; $3^3=27$; $3^4=81$; and $3^9=19683$. Again, if 64 is the base, the logarithm of 8 is $\frac{1}{2}$, or .5, since $64^{\frac{1}{2}}$, or $64^{.5}=8$; *i. e.*, $\frac{1}{2}$, or .5 is the exponent by which 64, the base, is to be affected in order to produce the number 8. So also, 64 being the base, $\frac{1}{3}$, or .333+ is the logarithm of 4, since $64^{\frac{1}{3}}$, or $64^{.333+}=4$; *i. e.*, $\frac{1}{3}$, or .333+ is the exponent by which 64, the base, is to be affected in order to produce the number 4. Once more, since $64^{\frac{2}{3}}$, or $64^{666+}=16$, $\frac{2}{3}$, or .666+ is the logarithm of 16, if the base is 64. Finally, $64^{-\frac{1}{2}}$, or $64^{-.5}=\frac{1}{8}$, or .125 ; hence $-\frac{1}{2}$, or $-.5$, is the logarithm of $\frac{1}{8}$, or .125, when the base is 64. In like manner, with the same base, $-\frac{1}{3}$, or $-.333+$ is the logarithm of $\frac{1}{4}$, or .25.

EXAMPLES.

Ex. 1. If 2 is the base, what is the logarithm of 4? of 8? of 32? of 128? of 1024?

Model Solution.

7 is the logarithm of 128, if 2 is the base, since 7 is the exponent by which 2 is to be affected in order to produce the number 128.

2. If 5 is the base, what is the logarithm of 625? of 15625? of 125? of 25?

3. If 10 is the base, what is the logarithm of 100? of 1,000? of 10,000? of 10,000,000?

4. If 2 is the base, what is the logarithm of $\frac{1}{4}$, or .25? of $\frac{1}{8}$, or .125? of $\frac{1}{32}$, or .03125? *Ans.* to the last, —5.

5. If 8 is the base, of what number is $\frac{2}{3}$, or .666+ the logarithm? of what number is $\frac{4}{3}$ or 1.333+ the logarithm? of what number is 2 the logarithm? of what number is $2\frac{1}{3}$, or 2.333+? of what number $3\frac{2}{3}$, or 3.666+?

Ans. to the last, 2048.

Scholium.—Since any number with 0 for its exponent is 1, the logarithm of 1 is 0, in all systems. Thus $10^0 = 1$, whence 0 is the logarithm of 1, in a system in which the base is 10.

118. **A System of Logarithms** is a scheme by which all numbers can be represented, either exactly or approximately, by exponents by which a fixed number (the base) can be affected. Negative numbers can have no logarithms.

119. There are *Two Systems* of Logarithms in common use, called, respectively, the *Briggean* or *Common* System, and the *Napierian* or *Hyperbolic* System. The base of the former is 10, and of the latter 2.71828+.

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120. One of the most important uses of logarithms is to facilitate the multiplication, division, involution, and the extraction of roots of large numbers. These processes are performed upon the following principles:

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121. **Prop. 1.**—*The sum of the logarithms of two numbers is the logarithm of their product.*

Demonstration.—Let a be the base of the system. Let m and n be any two numbers whose logarithms are x and y respectively. Then by definition $a^x = m$, and $a^y = n$. Multiplying these equations together we have $a^{x+y} = mn$. Whence $x+y$ is the logarithm of mn. Q. E. D.

***122.* Prop. 2.**—*The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

Demonstration.—Let a be the base of the system, and m and n any two numbers whose logarithms are, respectively, x and y. Then by definition we have $a^x = m$, and $a^y = n$. Dividing, we have $a^{x-y} = \frac{m}{n}$. Whence $x-y$ is the logarithm of $\frac{m}{n}$. Q. E. D.

***123.* Prop. 3.**—*The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

Demonstration.—Let a be the base, and x the logarithm of m. Then $a^x = m$; and raising both to any power, as the zth, we have $a^{xz} = m^z$. Whence xz is the logarithm of the zth power of m. Q. E. D.

***124.* Prop. 4.**—*The logarithm of any root of a number is the logarithm of the number divided by the number expressing the degree of the root.*

Demonstration.—Let a be the base, and x the logarithm of m. Then $a^x = m$. Extracting the zth root we have $a^{\frac{x}{z}} = \sqrt[z]{m}$. Whence $\frac{x}{z}$ is the logarithm of $\sqrt[z]{m}$. Q. E. D.

---

125. In order to apply these principles practically, we need what is called a *Table of Logarithms*. That is, a table from which we can readily obtain the logarithm of any number, or the number corresponding to any logarithm.

We will, therefore, proceed to show how such a table can be computed. Though the method about to be given is not the most expeditious now known, it is, nevertheless, the one used when our tables were first computed.

***126.* Prob.—To compute the common logarithm of any decimal number.**

Demonstration.—1st. It is evident that it is necessary to compute the logarithms of *prime* numbers only, since the logarithm of a composite number is the sum of the logarithms of its factors (**121**).

2nd. To compute the logarithms of the series of prime numbers. In the first place we know that the logarithm of 1 is 0, since $10^0 = 1$. Also the logarithm of 10 is 1, since $10^1 = 10$. Now if we find the logarithm of 5, we can get the logarithm of 2 by subtracting the logarithm of 5 from log. 10.* If there be any number which is the logarithm of 5 it is evident it must lie between 0, which is log. 1, and 1, which is log. 10. Therefore starting with

$$10^0 = 1 \quad \text{and}$$
$$10^1 = 10 \quad \text{multiplying them together}$$

we have $\quad 10^1 = 10$

Extracting the square root, $10^{.5} = \sqrt{10} = 3.162277+$.

Again, as 5 lies between 10 and 3.162277 + its logarithm lies between 1 and .5. Multiplying the last two equations we have $10^{1.5} = 31.62277+$.

Extracting the square root, $10^{.75} = \sqrt{31.62277+} = 5.623413+$.

Again, $\quad 10^{.5} = 3.162277+$

and $\quad 10^{.75} = 5.623413+$

Multiplying, $\quad 10^{1.25} = 17.7827895914+$

Extracting the square root, $10^{.625} = \sqrt{17.7827895914+} = 4.216964+$.

Again, multiplying this last by $10^{.75} = 5.623413+$, as 5 lies between these numbers, and extracting the square root, we have $10^{.6875} = 4.869674+$. In each case the exponent of 10 is the logarithm of the number; thus .6875 is log. 4.869674+. Continuing this process to 22 operations (!) we have log. 5.000000+ = .698970+, which is sufficiently accurate for ordinary purposes.

Now log. 10 − log. 5 = log. 2 = 1 − .698970 = .301030.

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* This is the common abbreviation of "logarithm of 10," and should be read "logarithm of 10," not "log ten," which is grossly inelegant.

To find log. 3, we would take $10^{.5} = 3.162277+$, and $10^0 = 1$, and proceed as before.

Were it our purpose to find log. 11, the computation would be as follows:

$$\begin{array}{l}
10^1 = 10 \\
10^2 = 100 \\
\hline
10^3 = 1000 \\
10 \quad = 10^{1.5} = \sqrt{1000} = 31.62277+ \\
10^1 \qquad = \qquad 10 \\
\hline
10^{\frac{5}{2}} = 10^{2.5} = 316.2277+ \\
10^{1.25} = \sqrt{316.2277+} = 17.78278+ \\
10^1 \qquad = \qquad 10 \\
\hline
10^{2.25} = 177.8278+ \\
10^{1.125} = \sqrt{177.8278+} = 13.33521+ \\
10^1 \qquad = \qquad 10 \\
\hline
10^{2.125} = 133.3521+ \\
10^{1.0625} = \sqrt{133.3521+} = 11.54782- \\
10^1 \qquad = \qquad 10 \\
\hline
10^{2.0625} = 115.4782+ \\
10^{1.03125} = \sqrt{115.4782+} = 10.74607+ \\
10^{1.0625} = \qquad 11.54782- \\
\hline
10^{2.09375} = 124.09368+ \\
10^{1.046875} = \sqrt{124.09368+} = 11.13973+ \\
10^{1.03125} = \qquad 10.74607+ \\
\hline
10^{2.078125} = 119.70845+ \\
10^{1.0390625} = 10.94113+
\end{array}$$

Whence 1.0390625 is log. 10.94113; and proceeding with the computation, the logarithm of 11 may be found with sufficient accuracy.

Scholium.—The pupil will not fail to be impressed with an idea of the immense labor involved in computing a table of logarithms. The common tables give the logarithms of numbers from 1 to 10,000, with provision, as will be seen hereafter, for using them to find the logarithms of much larger numbers, with sufficient accuracy for practical purposes. One page of such a table is given. (Page 386.)

***127.* Prob.—To find the logarithm of a number from the table.**

N.	0	1	2	3	4	5	6	7	8	9	D.
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	..99	.236	.374	.511	.648	.785	.922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	...9	.143	.277	.411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
N.	0	1	2	3	4	5	6	7	8	9	D.

Solution.—Page 386 is one page of a table of logarithms giving the logarithms of numbers from 1 to 10,000 *directly*, and from which the logarithms of other numbers can also be found with little trouble. Thus, let it be required to find the logarithm of 325. Now, as the logarithm of 100 is 2, and of 1000 is 3, the logarithm of 325 must be between 2 and 3, *i. e.* 2 and a fraction. The fractional part is all that is given in the table, as the integral can be known by simple inspection. Looking in the table down the column marked N (numbers), we find 325, and opposite it in the column headed 0, we find 1883, but just above this we observe 51, which belongs to this logarithm and which is simply omitted to save space in the table, since it really belongs as a prefix to all the logarithms clear down to the number 332 where it is replaced by 52. Prefixing the 51 to the 1883, we have .511883 as the decimal part of the logarithm. Hence log. 325 is 2.511883. In like manner the logarithm of *any number represented by three figures* is found from the table.

To find the logarithm of a number represented by four figures. Let it be required to find the logarithm of 2936. Looking for 293 (the first three figures) in the column of numbers, and then passing to the right until reaching the column headed 6, the fourth figure, we find 7756, to which prefixing the figures 46, which belong to all the logarithms following them till some others are indicated, we have for the decimal part of the logarithm of 2936, .467756. But, as 3 is the logarithm of 1000, and 4 of 10,000, log. 2936 is 3 and this decimal, or log. $2936 = 3.467756$.

To find the logarithm of a number represented by *more than 4 figures.* Let it be required to find the logarithm of 2845672. Finding the decimal part of logarithm of the first 4 figures 2845, as before, we find it to be .454082. Now the logarithm of 2846 is 153 (millionths, really) more than that of 2845. Hence, assuming that if an increase of the number by 1000 makes an increase in its logarithm of 153, an increase of 672 in the number, will make an increase in the logarithm of $\frac{672}{1000}$ or .672 of 153, or 103, omitting lower orders, and adding this to .454082, we have .454185 as the decimal part of log. 2845672. The integral part is 6, since 2845672 lies between the 6th and 7th powers of 10. Hence log. $2845672 = 6.454185$. Q. E. D.

Scholium 1.—If in seeking the logarithm of any number any of the dots noticed in the table are passed, their places are to be filled with 0's, and the first two figures of the decimal of the logarithm taken from the 0 column in the line below. Thus log. 3166 is

3.500511. This arrangement of the table is a mere matter of convenience to save space.

Scholium 2.—The column marked D is called the column of Tabular Differences; and any number in it is the difference between the logarithms found in columns 4 and 5, which is usually the same as between any two consecutive logarithms in the same horizontal line. The assumption made in using this difference; viz., that the logarithms increase in the same ratio as the numbers, is only approximately true, but still is accurate enough for ordinary use.

128. The Integral Part of a logarithm is called the **Characteristic**, and the decimal part the **Mantissa.**

129. **Prop.**—*The Mantissa of a decimal fraction, or of a mixed number, is the same as the mantissa of the number considered as integral.*

Demonstration.—Above it was found that log. 2845672 = 6.454185. Now this means that $10^{6.454185} = 2845672$. Dividing by 10 successively we have

$10^{5.454185} = 284567.2$, or log. 284567.2 $= 5.454185$,
$10^{4.454185} = 28456.72$ or log. 28456.72 $= 4.454185$,
$10^{3.454185} = 2845.672$ or log. 2845.672 $= 3.454185$,
$10^{2.454185} = 284.5672$ or log. 284.5672 $= 2.454185$,
$10^{1.454185} = 28.45672$ or log. 28.45672 $= 1.454185$,
$10^{0.454185} = 2.845672$ or log. 2.845672 $= 0.454185$.

Now if we continue the operation of division, only writing 0.454185 −1, $\bar{1}.454185$, meaning by this that the characteristic is negative and the mantissa positive, and the subtraction not performed, we have

$10^{\bar{1}.454185} = .2845672$, or log. .2845672 $= \bar{1}.454185$,
$10^{\bar{2}.454185} = .02845672$, or log. .02845672 $= \bar{2}.454185$,
$10^{\bar{3}.454185} = .002845672$, or log. .002845672 $= \bar{3}.454185$,
etc. Q. E. D.

Scholium.—The characteristic of an integral number, or of a mixed integral number and decimal, is one less than the number of integral places, as will appear by comparing such numbers with the powers of 10, as is done in demonstrating (**126**). The characteristic of a number entirely decimal fractional, is negative, and one greater than the number of 0's immediately following the decimal point, as appears from the last demonstration, or as appears from the fact that $10^{-1} = \frac{1}{10} = .1$; $10^{-2} = \frac{1}{100} = .01$; $10^{-3} = \frac{1}{1000} = .001$; etc.

EXAMPLES.

Find the logarithms of the following numbers: 285; 3145; 29056; 30942; 298.026; 32.56; 2.864; .3205; .00317; .00000328.

Results, log. 298.026 = 2.474254; log. .00317 $=\bar{3}.501059$.

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130. Prob.—To find a number corresponding to a given logarithm.

Solution.—Let it be required to find the number corresponding to the logarithm 5.515264. Looking in the table for the *next less* mantissa, we find .515211, the number corresponding to which is 3275 (no account now being taken as to whether it is integral, fractional or mixed; as in any case the figures will be the same). Now, from the *tabular difference*, in column D, we find that an increase of 133 (millionths, really) upon this logarithm (.515211), would make an increase of 1 in the number, making it 3276. But the given logarithm is only 53 greater than this, hence it is assumed (though only approximately correct) that the increase of the number is $\frac{53}{133}$ of 1, or $53 \div 133 = .3984+$. This added (the figures annexed) to 3275, gives 32753984+. The characteristic, being 5, indicates that the number lies between the 5th and 6th powers of 10, and hence has 6 integral places. ∴ 5.515264 = log. 327539.84+.

EXAMPLES.

Find the numbers corresponding to the following logarithms: 3.467521; $\bar{2}.467521$; 0.467521; 4.520281; 1.520281; $\bar{4}.520281$; 0.520281; $\bar{1}.520281$; 2.490160; $\bar{2}.490160$; and 0.490160.

Results, 2.490160 = log. 309.1435+; $\bar{2}.490160$ = log. .030914+; 0.490160 = log. 3.091435+.

131. As logarithms are largely used to facilitate numerical computations, it is important that the student be able to take any formula representing such operations and write at once the equivalent logarithmic operations.

EXAMPLES.

Ex. **1.** If 28.035 : 3.2781 : : 3114.27 : x, what logarithmic operations will find x?

Suggestion.—The logarithm of the product of the means is the sum of their logarithms; and the logarithm of the quotient of this product divided by the first extreme, is the logarithm of said product minus the logarithm of the other extreme. $\therefore$ log. $x =$ log. 3.2781 + log. 3114.27 − log. 28.035 $= 0.515622 + 3.493356 - 1.447700 = 2.561278$. Having a table sufficiently extended, the number corresponding to this logarithm could be found, and would be the value of x.

2. Find the product of 23 14, by 5.062, knowing that log. 23.14 is 1.364363, log. 5.062 is 0.704322, and log. 117.1347 is 2.068685.

3. How is 287 raised to the 5th power by means of logarithms? How is the 5th root extracted?

4. Extract the 5th root of 31152784.1 by means of logarithms, knowing that log. 31152784.1 = 7.493497.

Suggestion.—Log. $\sqrt[5]{31152784.1} = \frac{1}{5}$ log. $31152784.1 = 1.498699$. The number, therefore, is 31.52+.

5. What is the cube root of 30? *Ans.*, 3.107+.

6. What is the cube root of .03?

Suggestion.—Log. $.03 = \bar{2}.477121$. Now to divide this by 3, we have to bear in mind that the characteristic *alone* is negative, *i. e.*, $\bar{2}.477121 = -2 + .477121$, or -1.522879. This divided by 3 gives $-.507626$, or $0 - .507626 = \bar{1}.492374$. But a more convenient method of effecting this division is to write for the -2, $-3+1$, whence we have for $\bar{2}.477121$, $-3 + 1.477121$, which divided by 3 gives $\bar{1}.492374$, nearly.

7. Divide $\bar{3}.261453$ by 2, by 4, by 5.

Last quotient, $\bar{1}.4522906$.

APPENDIX

SECTION I.

DIFFERENTIATION.

[This subject is inserted as the best method of reaching the demonstration of the Binomial Formula and the production of the Logarithmic Series. While it is equally simple, to say the least, with the old method, it is more direct, and gives the student nothing but what is of fundamental importance in subsequent mathematical work.]

132. In certain classes of problems and discussions the quantities involved are distinguished as *Constant* and *Variable.*

133. ***A Constant*** quantity is one which maintains the same value throughout the same discussion, and is represented in the notation by one of the leading letters of the alphabet.

134. ***Variable*** quantities are such as may assume in the same discussion any value within certain limits determined by the nature of the problem, and are represented by the final letters of the alphabet.

ILL.—If x is the radius of a circle and y is its area, $y = \pi x^2$, as we learn from Geometry, π being about 3.1416. Now if x, the radius,

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NOTE.—The preceding part of this volume furnishes a course in Algebra quite as full as will be found practicable or desirable in most high schools and academies, and is an adequate preparation for college. This appendix, selected from OLNEY'S UNIVERSITY ALGEBRA, is inserted for such of the above schools as desire a fuller course, and as adapting the book to the needs of many of our colleges which do not find it expedient to give as much time to this subject as is required to master the University Algebra. There is nothing in the ordinary college course which requires more Algebra than is found in this volume.

varies, y, the area, will vary; but π remains the same for all values of x and y. In this case x and y are the variables, and π is a constant.

Again, if y is the distance a body falls in time x, it is evident that the greater x is, the greater is y, *i. e.*, that when x varies y varies. We learn from Physics that $y = 16\frac{1}{12}x^2$, for comparatively small distances above the surface of the earth. In the expression $y = 16\frac{1}{12}x^2$, x and y are the variables, and $16\frac{1}{12}$ is a constant. $\therefore\ y \propto x^2$.

Once more, suppose we have $y^2 = 25x^3 - 3x^2 - 5$, as an expressed relation between x and y, and that this is the *only* relation which is required to exist between them; it is evident that we may give values to x *at pleasure*, and thus obtain corresponding values for y. Thus if $x = 1$, $y = \pm\sqrt{17}$, if $x = 2$, $y = \pm\sqrt{183}$, etc., etc. In such a case x and y are called variables. But we notice that if we give to x such a value as to make $3x^2 + 5 > 25x^3$ (as, for example, $\frac{1}{2}$, $\frac{1}{4}$, etc.), y will be imaginary. This is the kind of limitation referred to in our definition of variables.

135. SCH.—The pupil needs to guard against the notion that the terms *constant* and *variable* are synonyms for *known* and *unknown*, and the more so as the notation might lead him into this error. The quantities he has been accustomed to consider in Arithmetic and Elementary Algebra have *all* been constant. The distinction here made is a new one to him, and pertains to a new class of problems and discussions.

136. A Function is a quantity, or a mathematical expression, conceived as depending for its value upon some other quantity or quantities.

ILL.—A man's wages *for a given time* is a function of the amount received per day, or, in general, his wages is a function of the time he works and the amount he receives per day. In the expression $y = 16\frac{1}{12}x^2$ (***134***), second illustration, y is a function of x, *i. e.*, the space fallen through is a function of the time. The expression $2ax^2 - 3x + 5b$, or any expression containing x, may be spoken of as a function of x.

137. When we wish to indicate that one variable, as y, is a function of another, as x, and do not care to be more specific, we write $y = f(x)$, and read "y equals (or is) a function of x." This means nothing more than that y is equal to some expression containing the variable x, and which besides may contain any constants. If we wish to indicate several *different* expressions each of which contains x, we write $f(x)$, $\varphi(x)$, or $f'(x)$, etc., and read "the f function of x," "the φ function of x," or "the f' function of x."

ILL.—The expression $f(x)$ may stand for x^3-2x+5, or for $3(a^2-x^2)$, or for *any* expression containing x combined in any way with itself or with constants. But in the same discussion $f(x)$ will mean the same thing throughout. So again, if in a particular discussion we have a certain expression containing x (*e. g.*, $3x^2-ax+2ab$), it may be represented by $f(x)$, while some *other* function of x, *e. g.*, $5(a^3-x^3)+2x^2$, might be represented by $f'(x)$, or $\phi(x)$.

138. In equations expressing the relation betweeen two variables, as in $y^2=3ax^3-x^2$, it is customary to speak of one of the variables, as y, as a function of the other, x. Moreover, it is convenient to think of x as varying and thus producing change in y. When so considered, x is called the *Independent* and y the *Dependent* variable. Or we may speak of y as a function of the variable x.

139. An Infinitesimal is a quantity conceived under such a form, or law, as to be necessarily less than any assignable quantity.

Infinitesimals are the increments by which continuous number, or quantity (*8*), may be *conceived* to change value, or grow.

ILL.—*Time* affords a good illustration of continuous quantity, or number. Thus a period of time, as 5 hours, increases, or grows, to another period, as 7 hours, by infinitesimal increments, *i. e.*, not by hours, minutes, or even seconds, but by elements which are less than *any* assignable quantity.

140. Consecutive Values of a variable are values which differ from each other by less than any assignable quantity, *i. e.*, by an infinitesimal. Consecutive values of a *function* are values which correspond to consecutive values of its variable.

141. A Differential of a function, or variable, is the difference between two consecutive states of the function, or variable. It is the same as an infinitesimal.

ILL.—Resuming the illustration $y = 16\frac{1}{12}x^2$ (*134*), let x be thought of as some particular period of time (as 5 seconds), and y as the distance through which the body falls in that time. Also, let x' represent a period of time only infinitesimally greater than x, and y' the distance through which the body falls in time x'. Then x and x' are consecutive values of x, and y and y' are consecutive values of y. Again, the difference between x and x', as $x'-x$, is a differential of the variable x, and $y'-y$ is a differential of the function y.

142. Notation.—A differential of x is represented by writing the letter d before x, thus dx. Also, dy means, and is read "differential y."

CAUTION.—Do not read dx by naming the letters as you do ax; but read it "differential x." The d is not a factor, but an abbreviation for the word *differential.*

143. To Differentiate a function is to find an expression for the increment of the function due to an infinitesimal increment of the variable; or it is the process of finding the relation between the infinitesimal increment of the variable and the corresponding increment of the function.

RULES FOR DIFFERENTIATING.

***144.* Rule I.**—*To differentiate a single variable, simply write the letter d before it.*

This is merely doing what the notation requires. Thus, if x and x' are consecutive states of the variable x, *i. e.*, if x' is what x becomes when it has taken an infinitesimal increment, $x'-x$ is the differential of x, and is to be written dx. In like manner, $y'-y$ is to be written dy, y' and y being consecutive values.

145. Rule II.—*Constant factors or divisors appear in the differential the same as in the function.*

Dem.—Let us take the function $y = ax$, in which a is any constant, integral or fractional. Let x take an infinitesimal increment dx, becoming $x+dx$; and let dy be the corresponding * increment of y, so that when x becomes $x+dx$, y becomes $y+dy$. We then have

1st state of the function . . $y = ax$;
2d, or consecutive state . . $\underline{y+dy = a(x+dx) = ax+adx.}$
Subtracting the 1st from the 2d $dy = adx$,

which result being the difference between two consecutive states of the function, is its differential (*141*). Now a appears in the differential just as it was in the function. This would evidently be the same if a were a fraction, as $\frac{1}{m}$. We should then have, in like manner, $dy = \frac{1}{m}dx$, for the differential of $y = \frac{1}{m}x$. Q. E. D.

146. Rule III.—*Constant terms disappear in differentiating; or the differential of a constant is 0.*

Dem.—Let us take the function $y = ax+b$, in which a and b are constant. Let x take an infinitesimal increment and become $x+dx$; and let dy be the increment which y takes in consequence of this change in x, so that when x becomes $x+dx$, y becomes $y+dy$. We then have

1st state of the function . . $y = ax+b$;
2d, or consecutive state . . $\underline{y+dy = a(x+dx)+b = ax+adx+b.}$
Subtracting the 1st from the 2d $dy = adx$,

which being the difference between two consecutive states of the function, is its differential (*141*). Now from this differential the constant b has disappeared.

We may also say that as a constant retains the same value, there is no difference between its consecutive states (properly it has no consecutive states). Hence the differential of a constant may be spoken of as 0. Q. E. D.

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* The word "contemporaneous" is often used in this connection.

147. Rule IV.—*To differentiate the algebraic sum of several variables, differentiate each term separately and connect the differentials with the same signs as the terms.*

DEM.—Let $u = x + y - z$, u representing the algebraic sum of the variables x, y, and $-z$. Then is $du = dx + dy - dz$. For let dx, dy, and dz be infinitesimal increments of x, y, and z; and let du be the increment which u takes in consequence of the infinitesimal changes in x, y, and z. We then have

1st state of the function . . . . $u = x + y - z$;
2d, or consecutive state . . . . $u + du = x + dx + y + dy - (z + dz)$.
Or . . . . $\underline{u + du = x + dx + y + dy - z - dz.}$
Subtracting the 1st state from the 2d $du = dx + dy - dz$. Q. E. D.

148. Rule V.—*The differential of the product of two variables is the differential of the first into the second, plus the differential of the second into the first.*

DEM.—Let $u = xy$ be the first state of the function. The consecutive state is $u + du = (x + dx)(y + dy) = xy + ydx + xdy + dx \cdot dy$. Subtracting the 1st state from the consecutive state we have the differential, *i. e.*, $du = ydx + xdy + dx \cdot dy$. But, as $dx \cdot dy$ is the product of two infinitesimals, it is infinitely less than the other terms (ydx and xdy), and hence, having no value as compared with them, is to be dropped.* Therefore, $du = ydx + xdy$. Q. E. D.

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* It will doubtless appear to the pupil, at first, as if this gave a result only *approximately* correct. Such is not the fact. The result is *absolutely* correct. *No* error is introduced by dropping $dx \cdot dy$. In fact this term *must* be dropped according to the nature of infinitesimals. Notice that by definition a quantity which is infinitesimal with respect to another is one which has *no* assignable magnitude with reference to that other. Hence we must so treat it in our reasoning. Now $dx \cdot dy$ is an infinitesimal of an infinitesimal (*i. e.*, two infinitesimals multiplied together), and hence is infinitesimal with reference to ydx and xdy, and must be treated as having *no* assignable value with respect to them; that is, it must be dropped.

149. Rule VI.—*The differential of the product of several variables is the sum of the products of the differential of each into the product of all the others.*

DEM.—Let $u = xyz$; then $du = yzdx + xzdy + xydz$. For the 1st state of the function is $u = xyz$, and the 2d, or consecutive state, $u + du = (x + dx)(y + dy)(z + dz)$, or $u + du = xyz + yzdx + xzdy + xydz + xdydz + ydxdz + zdxdy + dxdydz$. Subtracting, and dropping all infinitesimals of infinitesimals (see preceding rule and foot-note), we have $du = yzdx + xzdy + xydz$.

In a similar manner the rule can be demonstrated for any number of variables. Q. E. D.

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150. Rule VII.—*The differential of a fraction having a variable numerator and denominator is the differential of the numerator multiplied by the denominator, minus the differential of the denominator multiplied by the numerator, divided by the square of the denominator.*

DEM.—Let $u = \frac{x}{y}$; then is $du = \frac{ydx - xdy}{y^2}$. For, clearing of fractions, $yu = x$. Differentiating this by Rule 5th, we have $udy + ydu = dx$. Substituting for u its value $\frac{x}{y}$, this becomes $\frac{xdy}{y} + ydu = dx$. Finding the value of du, we have $du = \frac{ydx - xdy}{y^2}$. Q. E. D.

151. COR.—*The differential of a fraction having a constant numerator and a variable denominator is the product of the numerator with its sign changed into the differential of the denominator, divided by the square of the denominator.*

Let $u = \frac{a}{y}$. Differentiating this by the rule and calling the differential of the constant (a) 0, we have $du = \frac{0 - ady}{y^2} = \frac{-ady}{y^2}$. Q. E. D.

152. Sch.—If the numerator is variable and the denominator constant, it falls under Rule 2.

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153. **Rule VIII.**—*The differential of a variable affected with an exponent is the continued product of the exponent, the variable with its exponent diminished by 1, and the differential of the variable.*

Dem.—1st. *When the exponent is a positive integer.* Let $y = x^m$, m being a positive integer; then $dy = mx^{m-1}dx$. For $y = x^m = x \, . \, x \, . \, x \, . \, x$. to m factors. Now, differentiating this by Rule 6, we have $dy =$ (xxx . . to $m-1$ factors) $dx +$ (xxx . . to $m-1$ factors) $dx +$ etc., to m terms; or $dy = x^{m-1}dx + x^{m-1}dx + x^{m-1}dx +$ etc., to m terms. Therefore $dy = mx^{m-1}dx$.

2d. *When the exponent is a positive fraction.* Let $y = x^{\frac{m}{n}}$, $\frac{m}{n}$ being a positive fraction; then $dy = \frac{m}{n}x^{\frac{m}{n}-1}dx$. For involving both members to the nth power we have $y^n = x^m$. Differentiating this as just shown, we have $ny^{n-1}dy = mx^{m-1}dx$. Now from $y = x^{\frac{m}{n}}$ we have $y^{n-1} = x^{\frac{mn-m}{n}}$. Substituting this in the last it becomes $nx^{\frac{mn-m}{n}}dy = mx^{m-1}dx$; whence $dy = \frac{m}{n}x^{m-1-\frac{mn-m}{n}}dx = \frac{m}{n}x^{\frac{m}{n}-1}dx$. Q. E. D.

3d. *When the exponent is negative.* Let $y = x^{-n}$, n being integral or fractional; then $dy = -nx^{-n-1}dx$. For $y = x^{-n} = \frac{1}{x^n}$, which differentiated by Rule 7, Cor., gives $dy = -\frac{nx^{n-1}dx}{x^{2n}} = -nx^{-n-1}dx$. Q. E. D.

EXAMPLES.

1. Differentiate $y = 3x^2 - 2x + 4$.

Solution.—The result is $dy = 6xdx - 2dx$. Which is thus obtained: By Rule 1, the differential of y is dy. To differentiate the second member we differentiate each term separately according to Rule 4. In differentiating $3x^2$, we observe that the factor 3 is retained in the differential, Rule 2, and the differential of x^2 is, by Rule 8, $2xdx$. Hence, the differential of $3x^2$ is $6xdx$. The differential of $-2x$ is $-2dx$. By Rule 3, the constant 4 disappears from the differential, or its differential is 0.

2. Differentiate $y = 2ax^2 + 4ax^3 - x + m$.

Result, $dy = 4axdx + 12ax^2dx - dx$.

3. Differentiate $y = 5bx^3 - 30x^3 + 4x$.

4. Differentiate $y = Ax^2 + Bx^3 + Cx^4$.

154. Sch.—It is desirable that the pupil not only become expert in writing out the differentials of such expressions as the above, but that he know what the operation signifies. Thus, suppose we have the equation $y = 5x$. This expresses a relation between x and y. Now, if x changes value, y must change also in order to keep the equation true. In this simple case it is easy to see that y must change 5 times as fast as x in order to keep the equation true. This is what differentiation shows. Thus, differentiating, we have $dy = 5dx$. That is, if x takes an infinitesimal increment, y takes an infinitesimal increment equal to 5 times that which x takes; or, in other words, y increases 5 times as fast as x.

Now let us take a case which is not so simple. Let $y = 3x^2 - 2x + 4$, and let it be required to find *the relative rate of change of* x *and* y. Differentiating, we have $dy = 6xdx - 2dx = (6x - 2)\,dx$. This shows that, if x takes an infinitesimal increment represented by dx, y takes one (represented by dy) which is $6x-2$ times as large; *i. e.*, that y increases $6x-2$ times as fast as x. Notice that in this case the relative rate of increase of x and of y depends on the value of x. Thus, when $x = 1$, y is increasing 4 times as fast as x; when $x = 2$, y is increasing 10 times as fast as x; when $x = 3$, y is increasing 16 times as fast as x; etc.

5. Differentiate $y = x^5 - x^3$, and explain the significance of the result as above. *Result*, $dy = (5x^4 - 3x^2)\,dx$.

6. In order to keep the relation $2y = 3x^2$ true as x varies, how must y vary in relation to x? What is the relative

rate of change when $x = 4$? When $x = 2$? When $x = 1$? When $x = \frac{1}{3}$? When $x = \frac{1}{6}$?

Answers. When $x = 4$, y increases 12 times as fast as x. When $x = \frac{1}{3}$, y increases at the same rate as x. In general y increases $3x$ times as fast as x. When x is less than $\frac{1}{3}$, y increases slower than x.

7 to 12. Differentiate the following: $u = \frac{2x^2}{3y}$; $u = \frac{x^2-1}{x^2+1}$; $y = x^{\frac{1}{2}}z^2$; $u = x^2y^3+6x$; $y = x^5-3x^4+4x^3-x^2+1$; and $y = \frac{1}{2}x^2-\frac{2}{3}x^3+x$.

13 to 17. Differentiate $y = (a^2+x^3)^5$; $y = (a+x^2)^{\frac{2}{3}}$; $y = (3x-2)^{\frac{3}{2}}$; $y = (2-x^2)^{-2}$; and $y = (1+x)^{-\frac{1}{2}}$.

SUG'S.—Such examples should be solved by considering the entire quantity within the parenthesis as the variable. This is evidently admissible, since any expression which contains a variable is variable when taken as a whole. Thus to differentiate $y = (a+x^2)^{\frac{2}{3}}$, we take the continued product of the exponent $(\frac{2}{3})$, the variable $(a+x^2)$ with its exponent diminished by 1, [*i. e.*, $(a+x^2)^{-\frac{1}{3}}$], and the differential of the variable (*i. e.* the differential of $a+x^2$, which is $2xdx$). This gives us $dy = \frac{2}{3}(a+x^2)^{-\frac{1}{3}}2xdx$, or $dy = \frac{4}{3}x(a+x^2)^{-\frac{1}{3}}dx = \frac{4xdx}{3\sqrt[3]{a+x^2}}$.

18 to 22. Differentiate $\frac{1}{1+x}$; $\frac{1}{(1+x)^2}$; $\frac{1}{(1+x)^3}$; $-m\frac{1}{(1+x)^5}$; and $-m\frac{1}{(1+x)^3}$.

23. In the expression $6x^3$, when x is greater than 1 does the function $(6x^3)$ change faster or slower than x? How, when x is less than $\frac{1}{2}$? What does the process of differentiating $6x^3$ signify?

Answer to the last. Finding the relative rate of change of $6x^3$ and of x, or finding what increment $6x^3$ takes when x takes the increment dx.

Or, in still other words, finding the difference between two consecutive states of $6x^3$, and hence the relation between an infinitesimal increment of x and the corresponding increment of $6x^3$.

INDETERMINATE COEFFICIENTS.

155. *Indeterminate Coefficients* are coefficients assumed in the demonstration of a theorem or the solution of a problem, whose values are not known at the outset, but are to be determined by subsequent processes.

156. *Prop.*—*If* $A+Bx+Cx^2+Dx^3+$ *etc.* $=A'+B'x+C'x^2+D'x^3+$ *etc., in which* x *is a variable and the coefficients* A, B, A′ B′, *etc., are constants, the coefficients of the like powers of* x *are equal to each other. That is,* $A=A'$ (*these being the coefficients of* x^0), $B=B'$, $C=C'$, *etc.*

DEM.—Since the equation is true for *any* value of x, it is true for $x=0$. Substituting this value, we have $A=A'$. Now as $A=A'$ are constant, they have the same values whatever the value assigned to x. Hence for *any* value of x, $A=A'$. Again, dropping A and A', we have $Bx+Cx^2+Dx^3+$ etc. $=B'x+C'x^2+D'x^3+$ etc., which is true for any value of x. Dividing by x, we obtain $B+Cx+Dx^2+$ etc. $=B'+C'x+D'x^2+$ etc., likewise true for any value of x. Making $x=0$, $B=B'$, as before. In this manner we may proceed, and show that $C=C'$, $D=D'$, etc. Q. E. D.

157. COR.—*If* $A+Bx+Cx^2+Dx^3+$ *etc.* $=0$, *is true for all values of* x, *each of the coefficients* A, B, C, *etc., is* 0.

For we may write $A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+$ etc. $=0+0x+0x^2+0x^3+0x^4+0x^5+$ etc. Whence by the proposition $A=0$, $B=0$, $C=0$, etc.

SECTION II.

DEMONSTRATION OF THE BINOMIAL FORMULA.

158. *Theorem.* —*Whatever the value of* m, i. e., *whether integral or fractional, positive or negative.*

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2$$

$$+\frac{m(m-1)(m-2)}{\underline{\lvert 3}}a^{m-3}x^3$$

$$+\frac{m(m-1)(m-2)(m-3)}{\underline{\lvert 4}}a^{m-4}x^4 + \text{etc.}$$

DEM.—Assume

$$(a+x)^m = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{ etc.} \qquad (1)$$

in which A, B, C, etc., are indeterminate coefficients independent of x (*i. e.* constants), and are to be determined. To determine these coefficients we proceed as follows:

Differentiating (1), we have

$$m(a+x)^{m-1}dx = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + Fx^4dx + \text{ etc.}$$

Dividing by dx, we have

$$m(a+x)^{m-1} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \text{ etc.} \qquad (2)$$

Differentiating (2) and dividing by dx, we have

$$m(m-1)(a+x)^{m-2} = 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + 4\cdot 5Fx^3 + \text{ etc.} \qquad (3)$$

Differentiating (3) and dividing by dx, we have

$$m(m-1)(m-2)(a+x)^{m-3} = 2\cdot 3D + 2\cdot 3\cdot 4Ex + 3\cdot 4\cdot 5Fx^2 + \text{ etc.} \quad (4)$$

Differentiating (4) and dividing by dx, we have

$$m(m-1)(m-2)(m-3)(a+x)^{m-4} = 2\cdot 3\cdot 4E + 2\cdot 3\cdot 4\cdot 5Fx + \text{ etc.} \quad (5)$$

Differentiating (5) and dividing by dx, we have

$$m(m-1)(m-2)(m-3)(m-4)(a+x)^{m-5} = 2 \cdot 3 \cdot 4 \cdot 5F + \text{etc.} \qquad (6)$$

We have now gone far enough to enable us to determine the coefficients A, B, C, D, E, and F, and doubtless to determine the law of the series.

As all the above equations are to be true for all values of x, and as the coefficients A, B, C, etc., are constants, *i. e.*, have the same values for one value of x as for another, if we can determine their values for one value of x, these will be their values in all cases. Now, making $x=0$, we have from (1) $A=1$; from (2), $B=m$; from (3), $C=\dfrac{m(m-1)}{\underline{|2}}$; from (4), $D=\dfrac{m(m-1)(m-2)}{\underline{|3}}$; from (5), $E=\dfrac{m(m-1)(m-2)(m-3)}{\underline{|4}}$; from (6), $F=\dfrac{m(m-1)(m-2)(m-3)(m-4)}{\underline{|5}}$.

These values substituted in (1) give

$$(a+x)^m = a^m + ma^m x^{m-1} + \frac{m(m-1)}{2} a^{m-2}x^2$$

$$+ \frac{m(m-1)(m-2)}{\underline{|3}} a^{m-3}x^3 + \frac{m(m-1)(m-2)(m-3)}{\underline{|4}} a^{m-4}x^4$$

$$+ \frac{m(m-1)(m-2)(m-3)(m-4)}{\underline{|5}} a^{m-5}x^5 + \text{etc.}$$

Which is the ***Binomial Formula.***

159. Cor. 1.—*The* n*th, or general term of the series is*

$$\frac{m(m-1)\,(m-2)\,.\,.\,.\,.\,.\,(m-n+2)}{n-1}\, x^{m-n+1}y^{n-1}$$

For we observe that the last factor in the numerator of the coefficient of any particular term is $m-$ the number of the term less 2, *i. e.*, for the nth term, $m-(n-2)$, or $m-n+2$; and the last factor in the denominator is the number of the term -1, *i, e.*, for the nth term, $n-1$. The exponent of x in any particular term is $m-$ the number of the term less 1, *i. e.*, for the nth term, $m-(n-1)$, or $m-n+1$; and the exponent of y in any term is one less than the number of the term, *i. e.*, for the nth term, $n-1$.

160. Def.—In a series the ***Scale of Relation*** is the relation which exists between any term or set of terms and the next term or set of terms.

161. Cor. 2.—*The scale of relation in the binomial series is* $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$, *since the* n*th term multiplied by this produces the* $(n+1)$*th term.*

This is readily seen by inspecting the series, or by writing the $(n+1)$th term and dividing it by the nth. Thus, substituting in the general term as given above, $n+1$ for n, we have

$$\frac{m(m-1)(m-2)\text{ - - - - - }(m-n+1)}{\lfloor n}x^{m-n}y^{n},$$

as the $(n+1)$th term. This divided by the nth, or preceding term, gives $\frac{m-n+1}{n}\frac{y}{x}$, or $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$.

SECTION III.

THE LOGARITHMIC SERIES.

162. The Modulus of a system of logarithms is a constant factor which depends upon the base of the system and characterizes the system.

163. Prop.—*The differential of the logarithm of a* *number is the differential of the number multiplied by the modulus of the system, divided by the number;*

Or, in the Napierian system, the modulus being 1, the differential of the logarithm of a number is the differential of the number divided by the number.

DEM.—Let x represent *any* number, *i. e.* be a variable, and n be a constant such that $y = x^n$. Then $\log y = n \log x$ (*123*). Differentiating $y = x^n$, we have $dy = nx^{n-1}dx$; whence

$$n = \frac{dy}{x^{n-1}dx} = \frac{dy}{\frac{x^n}{x}dx} = \frac{dy}{\frac{y}{x}dx} = \frac{\frac{dy}{y}}{\frac{dx}{x}}. \qquad (1)$$

Again, whatever the differentials of $\log y$ and $\log x$ are, n being a constant factor we shall have the differential of $\log y$ equal to n times the differential of $\log x$, which may be written

$$d(\log y) = n \cdot d(\log x), \text{ whence } n = \frac{d(\log y)}{d(\log x)} \qquad (2.)$$

Now equating the values of n as represented in (1) and (2), we have $\frac{d(\log y)}{d(\log x)} = \frac{\frac{dy}{y}}{\frac{dx}{x}}$. Whence $d(\log y)$ bears the same ratio to $\frac{dy}{y}$, as $d(\log x)$ does to $\frac{dx}{x}$. Let m be this ratio. Then $d(\log y) = \frac{mdy}{y}$, and $d(\log x) = \frac{mdx}{x}$.

We are now to show that m is constant and depends on the base of the system.

To do this, take $y = z^{n'}$, from which we can find as above $n' = \frac{d(\log y)}{d(\log z)} = \frac{\frac{dy}{y}}{\frac{dz}{z}}$. Now as m is the ratio of $d(\log y)$ to $\frac{dy}{y}$, it is also the ratio of $d(\log z)$ to $\frac{dz}{z}$; and $d(\log z) = \frac{mdz}{z}$. Thus we see that in any case in the same system the same ratio exists between the differential of the logarithm of a number and the differential of the number divided by the number. Therefore m is a constant factor.

Again $\frac{dy}{dx} = m\frac{1}{x}$ indicates the relative rate of change of a logarithm and its number. Now it is evident that the larger the base the slower the logarithm will change with reference to the number. (See examples under Art 117.) But the factor $\frac{1}{x}$ varies inversely in the *number;* hence m must vary with, or be a function of the base.*

---

164. Prob.—*To produce the logarithmic series.*

Solution.—The logarithmic series, which is the foundation of the usual method of computing logarithms, and of much of the theory of logarithms, is the development of log $(1+x)$. To develop log $(1+x)$, assume

$$\log(1+x) = A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{ etc.,} \qquad (1)$$

in which x is a variable, and A, B, C, etc., are constants.

Differentiating (1), we have

$$\frac{mdx\dagger}{1+x} = Bdx+2Cxdx+3Dx^2dx+4Ex^3dx+5Fx^4dx+\text{ etc.}$$

Dividing by dx,

$$\frac{m}{1+x} = B+2Cx+3Dx^2+4Ex^3+5Fx^4+\text{ etc.} \qquad (2)$$

Differentiating (2), and dividing by dx, we have

$$-m\frac{1}{(1+x)^2} = 2C+2\cdot3Dx+3\cdot4Ex^2+4\cdot5Fx^3+\text{ etc.} \qquad (3)$$

Differentiating (3), and dividing by 2 and by dx, we have

$$m\frac{1}{(1+x)^3} = 3D+3\cdot4Ex+2\cdot3\cdot5Fx^2+\text{ etc.} \qquad (4)$$

Differentiating (4), and dividing by 3 and dx, we have

$$-m\frac{1}{(1+x)^4} = 4E+4\cdot5Fx+\text{ etc.} \qquad (5)$$

---

* What the relation of the modulus to the base is, we are not now concerned to know; it will be determined hereafter.

† The number is $1+x$; hence the differential is m times the differential of $1+x$ divided by the number $1+x$.

Differentiating (5), and dividing by 4 and dx, we have

$$m\frac{1}{(1+x)^5} = 5F + \text{ etc.*} \qquad (6)$$

We have now gone far enough to enable us to determine the coefficients A, B, C, D, E, and F, and these will probably reveal the law of the series.

As all the above equations are to be true for all values of x, and as the coefficients A, B, C, etc., are constant, *i. e.*, have the same values for one value of x as for another, if we can determine their values for one value of x, these will be their values in all cases. Now, making $x=0$, we have, from (1), $A = \log 1 = 0$; from (2), $B = m$; from (3), $C = -\frac{1}{2}m$; from (4), $D = \frac{1}{3}m$; from (5), $E = -\frac{1}{4}m$; from (6), $F = \frac{1}{5}m$. These values substituted in (1) give

$$\log(1+x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc}\right),$$

the law of which is evident. This is the *Logarithmic Series*, and should be fixed in memory.

SCH.—The Napierian system of logarithms is characterized by the modulus being 1 ($m=1$). Hence the *Napierian logarithmic series* is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$$

165. COR. 1.—*The logarithms of the same number in different systems are to each other as the moduli of those systems.*

This is evident from the general logarithmic series. Thus the logarithm of $1+x$ in a system whose modulus is m, is expressed

$$\log_m(1+x)\dagger = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right);$$

and the logarithm of the same number in a system whose modulus is m' is expressed

---

* Of course the student will observe what forms the succeeding terms in this and the other similar cases would have. Thus here we should have $5F + 5\cdot 6Gx + 3\cdot 5\cdot 7Hx^2 +$ etc.

† The subscripts m and m' are used to distinguish between the systems, as log $(1+x)$ is not the same in one system as in the other. Read $\log_m(1+x)$, "logarithm of $1+x$ in a system whose modulus is m," etc.

$$\log_{m'}(1+x) = m'\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right).$$

Now, as the number $(1+x)$ is, by hypothesis, the same in both cases, x is the same. Hence, dividing one equation by the other, we have

$$\frac{\log_m(1+x)}{\log_{m'}(1+x)} = \frac{m}{m'}.$$

166. Cor. 2.—*Having the logarithm of a number in the Napierian system, we have but to multiply it by the modulus of any other system to obtain the logarithm of the same number in the latter system.*

Or, the logarithm of a number in any system divided by the logarithm of the same number in the Napierian system, gives the modulus of the former system.

167. Prob.—*To adopt the Napierian logarithmic series to numerical computation so that it can be conveniently used for computing the logarithms of numbers.*

Dem.—That $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \text{etc.}$, is not in a practicable form for computing the logarithms of numbers will be evident if we make the attempt. Thus, suppose we wish to compute the logarithm of 3. Making $x = 2$, we have $\log(1+2) = \log 3 = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5} - \text{etc.}$, a series in which the terms are growing larger and larger (a diverging series).

We wish a series in which the terms will grow smaller as we extend it (a converging series). Then the farther we extend the series, the more nearly shall we approximate the logarithm sought. To obtain such a series, substitute $-x$ for x in the Napierian logarithmic series, and we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \text{etc.}$$

Subtracting this from the former series, we have

$$\log(1+x) - \log(1-x) = \log\left(\frac{1+x}{1-x}\right) = 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \text{etc.}\right).$$

Now put $x = \frac{1}{2z+1}$, whence $1+x = 1 + \frac{1}{2z+1} = \frac{2z+2}{2z+1}$, $1-x = \frac{2z}{2z+1}$, and $\frac{1+x}{1-x} = \frac{1+z}{z}$. Hence, as $\log\left(\frac{1+z}{z}\right) = \log(1+z) - \log z$, substituting, and transposing,

$$\log(1+z) = \log z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \text{etc.}\right) \quad (A)$$

This series converges quite rapidly, especially for large values of z, and is convenient for use in computing logarithms.

168. *Prob.*—*To compute the Napierian logarithms of the natural numbers* 1, 2, 3, 4, *etc., ad libitum.*

SOLUTION.—In the first place we remark that it is only necessary to compute the logarithms of *prime* numbers, since the logarithm of a composite number is equal to the sum of the logarithms of its factors (***121***).

Therefore beginning with 1, we know that log 1 = 0 (SCH., p. 382).

To compute the logarithm of 2, make $z = 1$, in series (A), and we have

$$\log(1+1) - \log 1 = \log 2 = 2\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \frac{1}{9.3^9} + \frac{1}{11.3^{11}} + \frac{1}{13.3^{13}} + \frac{1}{15.3^{15}} + \text{etc.}\right).$$

The numerical operations are conveniently performed as follows:

3	2 00000000		
9	.66666667	1	.66666667*
9	.07407407	3	.02469136
9	.00823045	5	.00164609
9	.00091449	7	.00013064
9	.00010161	9	.00001129
9	.00001129	11	.00000103
9	.00000125	13	.00000009
	.00000014	15	.00000001

$\therefore \log 2 = .69314718$ *

---

* Though the decimal part of a logarithm is generally not customary to annex the + sign.

Second. To find log 3, make $z = 2$, whence

$$\log 3 = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \frac{1}{7.5^7} + \frac{1}{9.5^9} + \text{etc.}\right).$$

COMPUTATION.

5	2.00000000		
25	.40000000	1	.40000000
25	.01600000	3	.00533333
25	.00064000	5	.00012800
25	.00002560	7	.00000366
	.00000102	9	.00000011

$.40546510$

$\log 2 = .69314718$

$\therefore \log 3 = 1.09861228$

Third. To find log 4.

$$\text{Log } 4 = 2 \log 2 = 2 \times .69314718 = 1.38629436.$$

Fourth. To find log 5. Let $x = 4$, whence

$$\log 5 = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3.9^3} + \frac{1}{5.9^5} + \frac{1}{7.9^7} + \text{etc.}\right).$$

COMPUTATION.

9	2.00000000		
81	.22222222	1	.22222222
81	.00274348	3	.00091449
81	.00003387	5	.00000677
	.00000042	7	.00000006

$.22314354$

$\log 4 = 1.38629436$

$\therefore \log 5 = 1.60943790$

In like manner we may proceed to compute the logarithms of the *prime* numbers from the formula, and obtain those of the composite numbers on the principle that the logarithm of the product equals the sum of the logarithms of the factors.

Thus, the Napierian logarithm of the base of the common system, 10, $= \log 5 + \log 2 = 2.30258508$.

169. Prop.—*The modulus of the common system is* .43429448+.

DEM.—Since the logarithm of a number, in any system, divided by the Napierian logarithm of the same number is equal to the modulus of that system (***166***), we have

$$\frac{\text{Com. log } 10}{\text{Nap. log } 10} = \text{modulus of common system.}$$

But com. log $10 = 1$, and Nap. log $10 = 2\ 30258508$, as found above. Hence,

$$\textit{Modulus of common system} = \frac{1}{2.30258508} = .43429448$$

170. Prop.—*The Napierian base is* 2.718281828.

DEM.—Let e represent the base of the Napierian system. Then by (***165***)

$$\text{com. log } e : \text{Nap. log } e :: .43429448 : 1.$$

But the logarithm of the base of a system, taken in that system is 1, since $a^1 = a$. Hence, Nap. log $e = 1$, and com. log $e = .43429448$. Now finding from a table of common logarithms the number corresponding to the logarithm .43429448, we have $e = 2.718281828$.

SECTION IV.

HIGHER EQUATIONS.

171. Since every equation with one unknown quantity, and real and rational coefficients, can be transformed into one of the form

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \dots L = 0, \qquad (1)$$

this will be taken as the typical numerical equation whose solution we shall seek in this and the succeeding sections;

and we shall frequently represent it by $f(x) = 0$, read "function x equals 0." The notation $f(x)$ signifies in general, as has been before explained (***137***), simply *any expression* involving x. Here we use it for this particular form of expression. We shall also use $f'(x)$ as the symbol for the first differential coefficient of this function.

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172. Prop.—*When an equation is reduced to the form* $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \dots . L = 0$, *the roots, with their signs changed, are factors of the absolute (known) term,* L.

[For demonstration see p. 363.]

173. Cor.—*If* a *is a root of* $f(x) = 0$, $f(x)$ *is divisible by* $x-a$; *and, conversely, if* $f(x)$ *is divisible by* $x-a$, a *is a root of* $f(x) = 0$.

Dem.—The first statement is demonstrated in the proposition, and the second is evident, since as $f(x)$ is divisible by $x-a$, let the quotient be $\phi(x)$; whence $(x-a)\phi(x) = 0$. Now $x = a$ will satisfy this equation, since it renders $x-a = 0$, and does not render $\phi(x)$ infinity, since by hypothesis x does not occur in the denominator.*

---

174. Prop.—*If the coefficients and absolute term in* $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \dots . L = 0$, *are all integers, the equation can have no fractional root.*

Dem.—Suppose in this equation $x = \frac{s}{t}$; $\frac{s}{t}$ being a simple fraction in its lowest terms. Substituting this value of x, we have

$$\frac{s^n}{t^n} + A\frac{s^{n-1}}{t^{n-1}} + B\frac{s^{n-2}}{t^{n-2}} + C\frac{s^{n-3}}{t^{n-3}} \dots . L = 0.$$

---

* Could there be a term of the form $\frac{[illegible]}{x-a}$ in $\phi(x)$, $x = a$ would render it ∞, and $(x-a)\phi(x)$ would be $0 \times \infty$, which is indeterminate, since $0 \times \infty = 0 \times \frac{1}{0} = \frac{0}{0}$.

Multiplying by t^{n-1} we obtain

$$\frac{s^n}{t} + As^{n-1} + Bts^{n-2} + Ct^2s^{n-3} \dots . Lt^{n-1} = 0.$$

Now, by hypothesis, all the terms except the first are integral, and the first is a simple fraction in its lowest terms, as by hypothesis s and t are prime to each other. But the sum of a simple fraction in its lowest terms and a series of integers cannot be 0. Therefore x cannot equal $\frac{s}{t}$, a fraction.

175. Sch.—This proposition does not preclude the possibility of *surd* roots in this form of equation. These are possible.

---

176. Prop.—*An equation* f (x) = 0 (***171***) *of the* n*th degree, has* n *roots (if it has any), and no more.*

Dem.—Let a be a root of $f(x) = 0$, which is of the nth degree. Dividing $f(x)$ by $x - a$ (***173***), we have $\phi(x) = 0$, an equation of the $(n - 1)$th degree.

Let b be a root of $\phi(x) = 0$, and divide $\phi(x)$ by $x - b$ (***173***). Call the quotient $\phi'(x)$, whence $\phi'(x) = 0$, an equation of the $(n - 2)$th degree. In this way the degree of the equation can be diminished by division until, after $n - 1$ divisions, there results $\phi^n(x)$ of the first degree, and the equation is $x - l = 0$. Therefore,

$$f(x) = (x - a)\,\phi(x) = (x - a)(x - b)\,\phi'(x) = (x - a)(x - b)(x - c)$$
$$\phi''(x) = (x - a)(x - b)(x - c) \dots . (x - l) = 0;$$

i. e., $f(x)$ is resolvable into n factors, of the form $x - m$.

Now, as $x = a$, or $x = b$, or $x =$ any one of the quantities $a, b, c \dots . l$, will render $f(x)$ equal to 0, each one of these will satisfy the equation $f(x) = 0$. Therefore this equation has n roots.

Again, since it is evident that we have resolved $f(x)$ into its *prime* factors with respect to x, there can be no other factor of the form $x - m$ in $f(x)$, hence no other root of $f(x) = 0$, and this whether m is equal to one or more of the roots $a, b, c \dots . n$, or not. Therefore $f(x) = 0$ has only n roots.

177. Cor. 1.—*The polynomial* $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \dots . L$, *or* f (x), $= (x - a)(x - b)(x - c) \dots . (x - l)$, *in which* a, b, c . . . . l *are the roots of* f (x) = 0.

178. Cor. 2.—*The equation* f (x) = 0 *may have* 2, 3, *or even* n *equal roots, as there is no inconsistency in supposing* a = b, a = b = c, *or* a = b = c = ... l, *in the above demonstration.*

179. Cor. 3.—*Imaginary roots enter into equations having only real coefficients, in conjugate pairs; that is, if* f (x) = 0 *has only real coefficients, if it has one root of the form* $\alpha + \beta\sqrt{-1}$, *it has another of the form* $\alpha - \beta\sqrt{-1}$; *or, if it has one of the form* $\beta\sqrt{-1}$, *it has another of the form* $-\beta\sqrt{-1}$.

This is evident, since only thus can $f(x) = (x - a)(x - b)(x - c) \ldots . (x - n)$; that is, if one root, a for example, is $\alpha - \beta\sqrt{-1}$, there must be another of the form $\alpha + \beta\sqrt{-1}$, in order that the product of these two factors shall not involve an imaginary. Thus, $[x - (\alpha + \beta\sqrt{-1}] \times [x - (\alpha - \beta\sqrt{-1}] = x^2 - 2\alpha x + (\alpha^2 + \beta^2)$, a real quantity. So also $(x - \beta\sqrt{-1})(x + \beta\sqrt{-1}) = x^2 + \beta^2$, a real quantity. But if the assumed imaginary roots be not in conjugate pairs, the product of the factors $(x - a)(x - b)(x - c) \ldots . (x - l)$ will involve imaginaries.

180. Cor. 4.—*Hence an equation of an odd degree must have at least one real root; but an equation of an even degree does not necessarily have any real root.*

Cor. 5.—*If an equation has a pair of imaginary roots, the known quantities entering into the equation may be so varied that the two imaginary roots shall first give place to two equal roots, and then these to two unequal roots.*

As shown above, imaginary roots arise from real quadratic factors in $f(x)$. Let $x^2 - 2ax + b$ be such a quadratic factor, whence $x^2 - 2ax + b = 0$ satisfies $f(x) = 0$, and $a \pm \sqrt{a^2 - b}$ are the corresponding roots of $f(x) = 0$. Now, if $b < a^2$, these roots are imaginary. If, however, b diminishes or a increases (or both change thus together), when $b = a^2$ the two imaginary roots disappear and we have in their place two real roots, each a. If the same change in a and b continues, so that a^2 becomes greater than b, the two real, equal roots in turn give place to two real, unequal roots. Now as a and b are functions of the known quantities of the equation $f(x) = 0$, such changes are evidently possible.

181. By means of the property exhibited in (*177*), produce the equations whose roots are given in the following examples:

1. Roots $1, -3, 4$.
2. Roots $\sqrt{2}, -\sqrt{2}, -1, 3$.
3. Roots $1, 2, 2, -3, 4$.
4. Roots $-3, 2+\sqrt{-1}, 2-\sqrt{-1}$.
5. Roots $3, -2, -2, -2, 1$.
6. Roots $\frac{2}{3}, \frac{1}{2}, -\frac{3}{5}$.
7. Roots $1 \pm \sqrt{-2}, 2 \pm \sqrt{-3}$.
8. Roots $1\frac{1}{2}, 2, \sqrt{3}, -\sqrt{3}$.
9. Roots $\sqrt{-2}, -\sqrt{-2}, \sqrt{5}, -\sqrt{5}$.
10. Roots $10, -13, \frac{1}{5}, 1$.
11. Roots $3-2\sqrt{3}, 3+2\sqrt{3}, 2-3\sqrt{-1}, 2+3\sqrt{-1}, 1, -1$.

182. Prop.*—If the equation* f (x) = 0 *has equal roots, the highest common divisor of* f (x) *and its differential coefficient,** f′ (x), *being put equal to* 0, *constitutes an equation which has for its roots these equal roots, and no other roots.*†

---

* The differential coefficient of a function is sometimes called its first derived polynomial.

† The student must not suppose that the roots of $f(x) = 0$, and its first differential coefficient $f'(x) = 0$, are necessarily alike. $f'(x) =$ a series of *terms* some of which may be + and some −, and which may destroy each other, so as to render $f'(x) = 0$, for other values of x than such as render $f(x) = 0$, and not necessarily for any which do render $f(x) = 0$, except the equal roots of the latter.

Dem.—Let a be one of the m equal roots of $f(x) = 0$, and let the other roots be $b, c \dots . l$; then $f(x) = (x-a)^m (x-b)(x-c) \dots . (x-l)$ (*177*). Differentiating (*149*) and dividing by dx, we have

$$f'(x) = m(x-a)^{m-1}(x-b)(x-c) \dots . (x-l) + (x-a)^m (x-c) \dots . (x-l) + \dots . + (x-a)^m (x-b)(x-c) \dots . + \text{etc.}$$

Now $(x-a)^{m-1}$ is evidently the highest common divisor of $f(x)$ and $f'(x)$, and $(x-a)^{m-1} = 0$ is an equation having a for its root, and having no other.

In a similar manner, if $f(x) = 0$ has two sets of equal roots, so that

$$f(x) = (x-a)^m (x-b)^r (x-c)(x-d) \dots . (x-l),$$

differentiating and dividing by dx, we have

$$f(x) = m(x-a)^{m-1}(x-b)^r(x-c)(x-d) \dots . (x-l) + (x-a)^m r(x-b)^{r-1}(x-c)(x-d) \dots . (x-l)$$

$$+ (x-a)_m (x-b)^r (x-d) \dots . (x-n) + (x-a)_m (x-b)^r (x-c) \dots . (x-l) + \dots . + (x-a)^m (x-b)^r (x-c)(x-d) \dots . + \text{etc.}$$

Now the highest common divisor of $f(x)$ and $f'(x)$ is evidently $(x-a)^{m-1}(x-b)^{r-1}$. Putting this equal to 0, we have $(x-a)^{m-1}(x-b)^{r-1} = 0$, an equation which is satisfied by $x = a$ and $x = b$, and by no other values.

Thus we may proceed in the case of any number of sets of equal roots.

183. Sch.—In searching for the equal roots of equations of high degree, it may be convenient to apply the process of the proposition several times. Thus, suppose that $f(x) = 0$ has m roots each equal to a, and r roots each equal to b. Then the highest common divisor of $f(x)$ and $f'(x)$ is of the form $(x-a)^{m-1}(x-b)^{r-1}$; whence $(x-a)^{m-1}(x-b)^{r-1} = 0$ is an equation having the equal roots sought. Therefore we can find the highest common divisor of $(x-a)^{m-1}(x-b)^{r-1}$, and its differential coefficient which will be of the form $(x-a)^{m-2}(x-b)^{r-2}$, and write $(x-a)^{m-2}(x-b)^{r-2} = 0$, as an equation containing the roots sought. This process continued will cause one of the factors $(x-a)$ or $(x-b)$ to disappear and leave $(x-a)^{m-r} = 0$, when $m > r$; $(x-b)^{r-m} = 0$, when $r > m$; or $(x-a)(x-b) = 0$, when $m = r$. From any one of these forms we can readily determine a root.

184. Prop.—*In an equation* f(x) = 0, f(x) *will change sign when* x *passes through any real root, if there is but one such root, or if there is an odd number of such roots; but if there is an* EVEN *number of such roots,* f(x) *will not change sign.*

Let $a, b, c \dots e$ be the roots of $f(x) = 0$, so that $f(x) = (x-a)(x-b)(x-c)\dots(x-e) = 0$ (*177*). Conceive x to start with some value less than the least root, and continuously increase till it becomes greater than the greatest root. As long as x is less than the least root, all the factors $x-a$, $x-b$, etc., are negative; but when x passes the value of the least root, the sign of the factor containing that root will become +,* and if there is no other equal root, this factor will be the only one which will change sign. Hence the product of the factors will change sign. But if there is an even number of roots, each equal to this, an even number of factors will change sign; whence there will be no change in the sign of the function. If, however, there is an odd number of equal roots, the passage of x through the value of this root will cause a change of sign in an odd number of factors, and hence will change the sign of the function.

Finally, as it is evident that the signs of the factors, and hence of the function, will remain the same while x passes from one root to another, and in all cases changes or does not change as above when x passes through a root, the proposition is established.

The following example will be found very instructive: $x^5+4x^4-14x^2-17x-6 = 0$. The least root of this equation is -3. When $x < -3$, $f(x)$ is $-$; when $x = -3$, $f(x) = 0$; when x passes -3, increasing, $f(x)$ changes from $-$ to $+$, and remains $+$ till $x = -1$, when it becomes 0, and changes sign as x passes -1, *notwithstanding they are equal roots.* But there is an *odd* number of such roots, viz., three. But in $x^3-14x^2+64x-96 = 0$, two equal roots of which are 4, if we substitute 2 we get $f(x) = -16$, and substituting 5, $f(x) = -1$, the function not changing sign, although a root has been passed.

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* Suppose c be the least root, and that c' is the next state of x greater than c; then $c'-c$ is +.

185. Prop.—*Changing the signs of the terms of an equation containing the odd powers of the unknown quantity changes the signs of the roots.*

DEM.—If $x = a$ satisfies the equation $x^6 - Ax^4 + Bx^3 - Cx + D = 0$, we have $a^6 - Aa^4 + Ba^3 - Ca + D = 0$. Now changing the signs of the terms containing the odd powers of x, we have $x^6 - Ax^4 - Bx^3 + Cx + D = 0$. This is satisfied by $x = -a$, if the former is by $x = a$. For, substituting $-a$ for x, we have $a^6 - Aa^4 + Ba^3 - Ca + D = 0$, the same as in the first instance.

186. COR.—*Changing the signs of the terms containing the even powers will answer equally well, since it amounts to the same thing; and if we are careful to put the equation in the complete form, changing the signs of the alternate terms will accomplish the purpose.*

ILL.—The *negative* roots of $x^3 - 7x + 6 = 0$, are the positive roots of $-x^3 + 7x + 6 = 0$, or of $x^3 - 7x - 6 = 0$ (0 being considered an *even* exponent); or, writing the equation $x^3 \pm 0x^2 - 7x + 6 = 0$, changing the signs of alternate terms, and then dropping the term with its coefficient 0, we obtain the same result.

Again, the negative roots of $x^6 - 7x^5 - 5x^4 + 8x^3 - 132x^2 + 508x - 240 = 0$, are the positive roots of $x^6 + 7x^5 - 5x^4 - 8x^3 - 132x^2 - 508x - 240 = 0$, or of $-x^6 - 7x^5 + 5x^4 + 8x^3 + 132x^2 + 508x + 240 = 0$.

187. Prob.—*To evaluate** f(x) *for any particular value of* x, *as* x = a, *more expeditiously than by direct substitution.*

SOLUTION.—As $f(x)$ is of the form $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \,.\,.\, L$, let it be required to evaluate $x^4 + Ax^3 + Bx^2 + Cx + D$ for $x = a$. Write the detached coefficients as below, with a at the right in the form of a divisor; thus

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* This means to find the value of. Thus, suppose we want to find the value of $x^6 - 5x^5 + 2x^4 - 3x^3 + 6x^3 - x - 12$, for $x = 5$. We might substitute 5 for x, of course, and accomplish the end. But there is a more expeditious way, as the solution of this problem will show.

$$\begin{array}{llllll}
1 & +A & +B & +C & +D & \vert \underline{a} \\
 & \underline{a} & \underline{a^2+Aa} & \underline{a^3+Aa^2+Ba} & \underline{a^4+Aa^3+Ba^2+Ca} & \\
 & a+A & a^2+Aa+B & a^3+Aa^2+Ba+C & a^4+Aa^3+Ba^2+Ca+D &
\end{array}$$

Having written the detached coefficients, and the quantity a for which $f(x)$ is to be evaluated as directed, multiply the first coefficient 1 by a, write the result under the second, and add, giving $a+A$. Multiply this sum by a, write the product under the third coefficient B, and add, giving a^2+Aa+B. In like manner continue till all the coefficients (including the absolute term, which is the coefficient of x^0) have been used, and we obtain $a^4+Aa^3+Ba^2+Ca+D$, which is the value of $f(x)$ for $x=a$.

Illustration.—To evaluate $x^6-5x^5+2x^4-3x^3+6^2-x-12$, for $x=5$:

$$\begin{array}{rrrrrrrl}
1 & -5 & +2 & -3 & +6 & -1 & -12 & \vert \underline{5} \\
 & \underline{5} & \underline{0} & \underline{10} & \underline{35} & \underline{205} & \underline{1020} & \\
 & 0 & 2 & 7 & 41 & 204 & 1008 &
\end{array}$$

Now 1008 is the value of $x^6-5x^5+2x^4-3x^3+6x^2-x-12$, for $x=5$; and it is easy to see that much labor is saved by this process.

We are now prepared for the solution of the following important practical problem:

188. Prob.—*To find the commensurable roots of numerical higher equations.*

The solution of this problem we will illustrate by practical examples.

EXAMPLES.

1. Find the commensurable roots of $x^5-2x^4-15x^3+8x^2+68x+48=0$, if it has any.

Solution.—By (*174*), if this equation has any commensurable roots they are integral;—it can have no fractional roots.

Again, by (*172*), the roots of this equation with their signs changed are factors of 48. Now, the integral factors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 48. Hence, if the equation has commensurable roots, they are some of these numbers, with either the + or − sign. We will, therefore, proceed to evaluate $f(x)$ (*i. e.*, in this case $x^5-2x^4-15x^3+8x^2+68x+48$), for $x=+1$, $x=-1$, $x=+2$, $x=-2$, etc., by (***187***), as follows:

$$\begin{array}{rrrrrrl}
1 & -2 & -15 & +8 & +68 & +48 & \underline{|\,+1} \\
 & 1 & -1 & -16 & -8 & 60 & \\
\hline
 & -1 & -16 & -8 & 60 & 108 &
\end{array}$$

Hence we see that for $x=+1$, $f(x)=108$, and $+1$ is not a root of $f(x)=0$. Trying $x=-1$, we have[*]

$$\begin{array}{rrrrrrl}
1 & -2 & -15 & +8 & +68 & +48 & \underline{|\,-1} \\
 & -1 & 3 & 12 & -20 & -48 & \\
\hline
 & -3 & -12 & 20 & 48 & 0 &
\end{array}$$

Thus we see that for $x=-1$, $f(x)=0$, and hence that -1 is a root of our equation.

We might now divide $f(x)$ by $x+1$ (***173***) and reduce the degree of the equation by unity. But it will be more expeditious to proceed with our trial. Let us therefore evaluate $f(x)$ for $x=+2$. Thus:

$$\begin{array}{rrrrrrl}
1 & -2 & -15 & +8 & +68 & +48 & \underline{|\,+2} \\
 & 2 & 0 & -30 & -44 & +48 & \\
\hline
 & 0 & -15 & -22 & 24 & 96 &
\end{array}$$

Hence for $x=+2$, $f(x)=96$, and $+2$ is not a root. Trying $x=-2$, we have

$$\begin{array}{rrrrrrl}
1 & -2 & -15 & +8 & +68 & +48 & \underline{|\,-2} \\
 & -2 & 8 & 14 & -44 & -48 & \\
\hline
 & -4 & -7 & 22 & 24 & 0 &
\end{array}$$

Hence for $x=-2$, $f(x)=0$, and -2 is a root. Trying $x=+3$, we have

$$\begin{array}{rrrrrrl}
1 & -2 & -15 & +8 & +68 & +48 & \underline{|\,+3} \\
 & 3 & 3 & -36 & -84 & -48 & \\
\hline
 & 1 & -12 & -28 & -16 & 0 &
\end{array}$$

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[*] Of course it is not necessary to retain the + sign, as we have done in the preceding operations: it has been done simply for emphasis.

Hence for $x = +3$, $f(x) = 0$, and $+3$ is a root. Trying $x = -3$, we have

$$\begin{array}{rrrrrr|r} 1 & -2 & -15 & +8 & +68 & +48 & -3 \\ & -3 & 15 & 0 & -24 & -132 & \\ \hline & -5 & 0 & 8 & 44 & -84 & \end{array}$$

Hence for $x = -3$, $f(x) = -84$, and -3 is not a root. Trying $x = 4$, we have

$$\begin{array}{rrrrrr|r} 1 & -2 & -15 & +8 & +68 & +48 & 4 \\ & 4 & 8 & -28 & -80 & -48 & \\ \hline & 2 & -7 & -20 & -12 & 0 & \end{array}$$

Hence for $x = 4$, $f(x) = 0$, and 4 is a root.

We have now found four of the roots, viz., -1, -2, 3, and 4. Their product with their signs changed is 24. Hence, by (*172*) $48 \div 24 = 2$ is the other root with its sign changed, *i. e.*, there are *two* roots -2.

That our equation had equal roots could have been ascertained by the principle in (*182*); but as the process of finding the H. C. D. is tedious, it is generally best to avoid it in practice.

to 11. Find the roots of the following:

2. $x^4 - x^3 - 39x^2 + 24x + 180 = 0;$
3. $x^3 + 5x^2 - 9x - 45 = 0;$
4. $x^3 + 2x^2 - 23x - 60 = 0;$
5. $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0;$
6. $x^3 - 8x^2 + 13x - 6 = 0;$
7. $x^4 - 11x^2 + 18x - 8 = 0;$[*]
8. $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 = 0;$
9. $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0;$
10. $x^4 - 45x^2 - 40x + 84 = 0;$
11. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$

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[*] In order to apply the process of evaluation, the coefficients of the missing powers must be supplied. Thus we have $1 + 0 - 11 + 18 - 8$.

12 to 17. Apply the process for finding equal roots (***192, 183***) to the following:

12. $x^3+8x^2+20x+16=0$;
13. $x^3-x^2-8x+12=0$;
14. $x^3-5x^2-8x+48=0$;
15. $x^4-11x^2+18x-8=0$;
16. $x^4+13x^3+33x^2+31x+10=0$;
17. $x^5-13x^4+67x^3-171x^2+216x-108=0$.

18 to 24. Having found all but two of the roots of each of the following by (***187***), reduce the equation to a quadratic by (***173***), and from this quadratic find the remaining roots:

18. $x^3-6x^2+10x-8=0$;
19. $x^4-4x^3-8x+32=0$;
20. $x^3-3x^2+x+2=0$;
21. $x^4-6x^3+24x-16=0$;
22. $x^4-12x^3+50x^2-84x+49=0$;[*]
23. $x^4-9x^3+17x^2+27x-60=0$;
24. $x^5-4x^4-16x^3+112x^2-208x+128=0$;

25. $2x^3-3x^2+2x-3=0$;[†]
26. $3x^3-2x^2-6x+4=0$;
27. $8x^3-26x^2+11x+10=0$;
28. $x^4-\frac{1}{2}x+\frac{3}{16}=0$; (Look out for equal roots.)
29. $x^4-6x^3+9\frac{1}{2}x^2-3x+4\frac{1}{2}=0$.

---

[*] Apply the method for finding equal roots.

[†] We have $x^3-\frac{3}{2}x^2+x-\frac{3}{2}=0$. Put $x=\frac{y}{k}$, whence $\frac{y^3}{k^3}-\frac{3}{2k^2}y^2+\frac{1}{k}y-\frac{3}{2}=0$, or $y^3-\frac{3k}{2}y^2+k^2y-\frac{3k^3}{2}=0$. If now $k=2$, we have $y^3-3y^2+4y-12=0$, which can be solved as before, for one value of y, and the equation then reduced to a quadratic and solved for the other values. Finally, remembering that $x=\frac{1}{2}y$, we have the values of x required.

SECTION V.

DISCUSSION, OR INTERPRETATION, OF EQUATIONS.

189. *To Discuss, or Interpret, an Equation or an Algebraic Expression,* is to determine its significance for the various values, absolute or relative, which may be attributed to the quantities entering into it, with special reference to noting any changes of value which give changes in the general significance.

Such discussions may be divided into two classes: 1st. The discussion of equations or expressions with reference to their constants; and 2d. The discussion of equations or expressions with reference to their variables.

The following principles are of constant use in such discussions:

190. *Prop.*—*A fraction, when compared with a finite quantity, becomes:*

1. *Equal to* 0, *when its numerator is* 0 *and its denominator finite, and when its numerator is finite and its denominator* ∞.

2. *Equal to* ∞, *when its numerator is finite and its denominator* 0, *and when its numerator is* ∞ *and its denominator finite.*

3. *It assumes an indeterminate form when numerator and denominator are both* 0, *and when they are both* ∞.*

DEM.—These facts appear when we consider that the value of a fraction depends upon the relative magnitudes of numerator and denominator.

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* By this it is meant that $\frac{0}{0}$ and $\frac{\infty}{\infty}$ *may* have a variety of values, not that they necessarily *do* have.

1. Let a be any constant and x a variable, then the fraction $\frac{x}{a}$ diminishes as x diminishes, and becomes 0 when x is 0. Again, the fraction $\frac{a}{x}$ diminishes as x increases, and when x becomes ∞, *i. e.*, greater than any assignable magnitude, $\frac{a}{x}$ becomes less than any assignable magnitude or infinitesimal, and is to be regarded as 0 in comparison with finite quantities. (See ***139*** and ***148***, DEM., and foot-note.)

2. As x increases, the fraction $\frac{x}{a}$ increases, and hence when x becomes infinite, the value of the fraction is infinite. Also, as x diminishes, the value of $\frac{a}{x}$ increases; hence, when x becomes infinitely small, or 0, the value of the fraction exceeds any assignable limits, and is therefore ∞.

3. Finally, if x and y are variables, $\frac{x}{y}$ *diminishes* as x diminishes, and *increases* as y diminishes. What then does it become when $x=0$, and $y = 0$? *i. e.*, what is the value of $\frac{0}{0}$? Simple arithmetic would lead us to suppose that $\frac{0}{0}$ was *absolutely* indeterminate, *i. e.*, that it might have any value whatever assigned to it, for $\frac{0}{0} = 5$, since $0 = 5 \times 0 = 0$; $\frac{0}{0} = 7$, since $0 = 7 \times 0 = 0$, etc. But a closer inspection will enable us to see that the symbol $\frac{0}{0}$ is not necessarily indeterminate, or rather that the expression which takes this form for particular values of its components, has not necessarily an indefinite number of values for these values of its components. Thus, what the value of $\frac{x}{y}$ will be when x and y each diminish to 0 will evidently depend upon the relative values of x and y at first, and which diminishes the faster. Suppose, for example, that $y = 5x$; then $y = \frac{x}{5x}$. Now, suppose x to diminish; the denominator will diminish 5 times as fast as the numerator, and whatever the value of x, the value of the frac,

tion will be $\frac{1}{7}$. So if $y = 7x$, $\frac{x}{y} = \frac{x}{7x}$, which is $\frac{1}{7}$ for any value of x. Hence, when $x = 0$, and $y = 0$, we have $\frac{x}{y} = \frac{0}{0} = \frac{x}{5x} = \frac{1}{5}$, or $\frac{x}{y} = \frac{0}{0} = \frac{x}{7x} = \frac{1}{7}$, or $\frac{x}{y} = \frac{0}{0} =$ any other value depending upon the relative values of x and y. So, also, if $x = \infty$, and $y = \infty$, $\frac{x}{y} = \frac{\infty}{\infty}$; but if $y = 6x$, we have $\frac{x}{y} = \frac{\infty}{\infty} = \frac{x}{6x} = \frac{1}{6}$. And so if $y = 10x$, we have $\frac{x}{y} = \frac{\infty}{\infty} = \frac{x}{10x} = \frac{1}{10}$. Thus we see that the *mere* fact that numerator and denominator become 0, or become ∞, does not determine the value of the fraction, *i. e.*, gives it an indeterminate form.

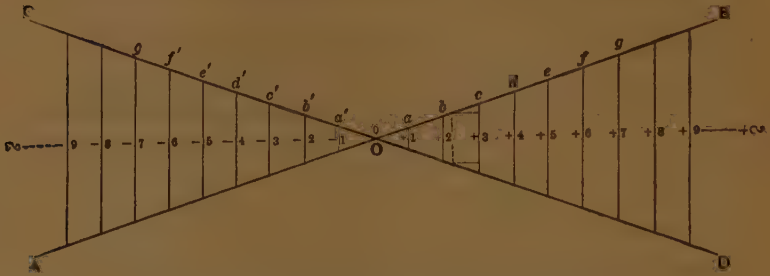
191. *A Real Number or Quantity* is one which may be conceived as lying somewhere in the series of numbers or quantities between $-\infty$ and $+\infty$ inclusive.

ILL.—Thus, if we conceive a series of numbers varying both ways from 0, *i. e.*, positively and negatively to ∞, we have

$$-\infty \ldots . -4, -3, -2, -1, 0, +1, +2, +3, +4, \ldots . +\infty.$$

Now a *real number* is one which may be conceived as situated somewhere within these limits; it may be +, −, integral, fractional, commensurable, or incommensurable. Thus, +15624 and −15624 will evidently be found in this series. $+\frac{17}{3}$ may be conceived as somewhere between +5 and +6, though *its exact locality* could not be fixed by the arithmetical conception of discontinuous number. So, also, $-\frac{17}{3}$ is somewhere between −5 and −6. Again, $+\sqrt{5}$ is somewhere between +2 and +3, though, as above, we cannot locate it exactly by the arithmetical conception.

The following *Geometrical Illustration* is more complete than the arithmetical. Thus, let two indefinite lines, as **CD** and **AB**, intersect (cross) each other, as at **O**. Now let parallel, equidistant lines be drawn between them. Call the one at a +1, that at b will be +2, at c +3, etc. So, also, the line at a' being −1, that at b' will be −2, at c' −3, etc. Now conceive one of these lines to start from an infinite distance at the left and move toward the right. When at an infinite distance to the left of **O** its value would be $-\infty$, and in passing to **O** it would pass through *all possible negative values.* In passing **O** it becomes 0 at **O**, changes sign to + as it passes, and moving on to infinity to the right, passes through *all possible positive values.* Hence we see how *all* real values are embraced between $-\infty$ and $+\infty$, inclusive.*



192. An Imaginary Number or Quantity is one which cannot be conceived as lying anywhere between the limits of $-\infty$ and $+\infty$, as explained above. The *algebraic form* of such a quantity is an expression involving an even root of a negative quantity.†

EXAMPLES.

1. What are the values of x and y in the expressions $x = \dfrac{b' - b}{a - a'}$, $y = \dfrac{ab' - a'b}{a - a'}$, when $b = b'$ and a and a' are

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* For example, the student who is acquainted with the elements of geometry knows how to construct a line which is exactly equal to $\sqrt{5}$ (GEOM., PART I, *110*). This line he can locate between +2 and +3, and also between −2 and −3, since $\sqrt{5}$ is both + and −.

† Transcendental functions afford other forms of imaginary expressions; for example, $\sin^{-1} 2$, $\sec^{-1} \frac{1}{2}$, $\log(-120)$, $\log(-m)$, etc. But our limits forbid the consideration of the interpretation of imaginaries, except in the most restricted sense, as indicating incompatibility with the arithmetical sense of the problem.

unequal? When $b=b'$ and $a=a'$? When $a=a'$ and b and b' are unequal? What are the *signs* of x and y when $b>b$ and $a>a'$, the essential signs of a, a', b, and b' being $+$? When $b>b'$ and $a<a'$? If a' and b are essentially negative, and $a=a'$, and $b=b'$, what are the values of x and y? If a' and b' are each 0?

2. What *general* relation between a and a' renders $\frac{a'-a}{1+aa'}=0$? What renders it ∞?

Solution.—To render $\frac{a'-a}{1+aa'}=0$, we must have $a'-a=0$, and $1+aa'$ finite or infinite; or else we must have $1+aa'=\infty$, while $a'-a$ is finite or 0 (***190***). Now $a'-a=0$ gives $a'=a$; whence, $\frac{a'-a}{1+aa'}=\frac{0}{1+a^2}$, which is 0 for any value of a, finite or infinite. Hence the relation $a'=a$ fulfills the first requirement. Let us now see if $1+aa'=\infty$ will also fulfill this requirement. This gives $aa'=\infty$, since subtracting 1 from ∞ would not make it other than ∞. Thus we have $a'=\frac{\infty}{a}$. Hence, for all finite values of a (including 0) a' is ∞, and $\frac{a'-a}{1+aa'}=\frac{a'}{aa'}$[*] $=\frac{1}{a}$; which can only be 0 when $a=\infty$. Therefore the *particular* values $a'=\infty=a=\infty$, render $\frac{a'-a}{1+aa'}=0$; but no *general* values do, unless $a=a'$.

Again, in order that $\frac{a'-a}{1+aa'}=\infty$, we must have $1+aa'=0$, and $a'-a$ finite or infinite; or else we must have $a'-a=\infty$, and $1+aa'$ finite or 0. Now $1+aa'=0$ gives

$$a=-\frac{1}{a'};\quad \frac{a'-a}{1+aa'}=\frac{a'+\frac{1}{a'}}{1-\frac{a'}{a'}}=\frac{a'^2+1}{a'-a'}=\frac{a'^2+1}{0}=\infty,$$

---

[*] This reduction is made by dropping a and 1, since the subtraction of a finite from an infinite, or the addition of a finite to an infinite, does not change the character of the infinite. Thus, in this case, to assume that dropping a and 1 affected the relation between numerator and denominator, would be to assign to a and 1 some values with respect to the infinite a'. But this is contrary to the definition of an infinite.

for any value of a', finite or infinite. Therefore the general relation $a=-\frac{1}{a'}$ between a and a' renders $\frac{a'-a}{1+aa'}=\infty$.* Let us now see if the relation $a'-a=\infty$ will do the same. Now if $a'-a=\infty$, one or the other (a' or a) must be ∞. Let $a'=\infty$. We then have $\frac{a'-a}{1+aa'}=\frac{a'}{aa'}=\frac{1}{a}$, which can only be ∞ when $a=0$. Hence the *particular* values $a'=\infty$ and $a=0$ render $\frac{a'-a}{1+aa'}=\infty$, but no general values meet the requirement unless $a=-\frac{1}{a'}$.

3. What general relation between a and a' renders $\frac{1-aa'}{a'+a}=0$? What renders it ∞?

4. In the expression $y=-2x+4\pm\sqrt{x^2-4x-5}$, how many values has y, *in general*, for any particular value of x? For what value or values of x has y but one value? For what values of x is y real? For what imaginary? For what values of x is y positive? For what negative?

SOLUTION.—Writing the expression thus,

$$y=-(2x-4)\pm\sqrt{x^2-4x-5},$$

we see that the value of y is made up of two parts, viz., a rational part $-(2x-4)$, and a radical part $\sqrt{x^2-4x-5}$. But the radical part may be taken with either the + or the — sign. Hence, *in general*, for any particular value of x *there are two values of* y. 2d. But if such a value is given to x as to render the radical part 0, for this value of x, y will have but one value, viz., the rational part. But the condition $\sqrt{x^2-4x-5}=0$ gives $x=5$ and -1. Thus, for $x=5$, $y=-6$, but *one* value; and for $x=-1$, $y=+6$, also but *one* value. 3d. To ascertain for what values of x, y is *real*, we observe that y is real when x^2-4x-5 is positive, and imaginary when x^2-4x-5 is negative. Now for x positive, $x^2-(4x+5)$ is + when $x^2>4x+5$; and for x negative, we have x^2+4x-5, which is positive when $x^2+4x>5$. The former inequality gives $x^2-4x+4>9$, or $x>5$; and the latter gives $x^2+4x+4>9$, or $x>1$. Hence for positive values of x greater than 5, y is real, and for negative values of x numerically greater than 1, y is real. The 4th inquiry is answered by this: y is imaginary for all values of x between -1 and $+5$. 5th. To ascertain what + values of x render y +, and what —, we observe that $-(2x-4)\pm\sqrt{x^2-4x-5}$ can only be + when the + sign of the radical part is taken and when $\sqrt{x^2-4x-5}>2x-4$. This gives $x<2\pm\sqrt{-3}$, *i. e.*, an imaginary quantity. Hence y is never + for x +. Taking the negative sign of the radical, we see that both parts of the value of y are —, and consequently y is real and negative for all + values of x which render y real, *i. e.*, for values greater than 5. Finally, for x — we have $y=2x+4\pm\sqrt{x^2+4x-5}$. Now when we take the + sign of the radical, both parts are + ; hence this value of y is always plus. When we take the — sign of the radical, y is negative if $2x+4<\sqrt{x^2+4x-5}$. But this gives $x<-2\pm\sqrt{-3}$. Hence y is never negative for any negative value of x. Therefore both values of y are positive and real for all negative values of x numerically greater than 1.

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* It is to be observed that the relation $a=-\frac{1}{a'}$ requires that a and a' have different essential signs; while the relation $a'=a$ requires that they have the same essential signs.

5 to 22. Discuss as above the values of y in the following; *i. e.*, 1st. Show how many values y has *in general*, and whether they are equal or unequal; 2d. For what particular value or values of x, y has but one value; 3d. For what values of x, y is real, and for what imaginary; 4th. For what values of x, y is +, and for what —; 5th. Also determine what values of x render y infinite:

5. $y^2+2xy-2x^2-4y-x+10=0$;*
6. $y^2-2xy+2x^2-2y+2x=0$;
7. $y^2+2xy+x^2-6y+9=0$;
8. $y^2+2xy+3x^2-4x=0$;
9. $y^2-2xy+3x^2+2y-4x-3=0$;
10. $y^2+2xy-3x^2-4x=0$;
11. $y^2-2xy+x^2+x=0$;

---

* In all cases solve the equation for y in the first place. In this example

$$y=-x+2\pm\sqrt{3x^2-3x-6}.$$

12. $y^2-2xy+x^2-4y+x+4=0$;
13. $y^2-2xy+x^2+2y+1=0$;
14. $y^2-2x^2-2y+6x-3=0$;
15. $y^2-2xy-3x^2-2y+7x-1=0$;
16. $y^2-2xy-2=0$;
17. $y^2-2xy+2y+4x-8=0$;
18. $4y^2+4x^2+2y-3x+12=0$;
19. $3y^2-8x^2=12$;
20. $12y^2+4x^2=20$;
21. $x^2+y^2=16$;
22. $x^2-y^2=20$.

193. Arithmetical Interpretations of Negative and Imaginary Solutions.

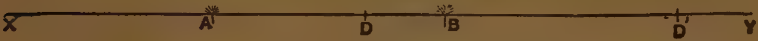
1. A is 20 years old, and B 16. When will A be twice as old as B?

SUG'S.—We have $20+x=2(16+x)$; whence $x=-12$. The arithmetical interpretation of this result is that A *will never* be twice as old as B, but that he *was* twice as old 12 years ago, *i. e.*, when he was 8 and B 4.

2. A is a years old, and B b. When will A be n times as old as B? For $n>1$ what are the possible relative values of a and b consistently with the arithmetical sense of the problem? Interpret for $a>nb$, $a=nb$, $a<nb$ when $n>1$. Also for $n=1$, $a>nb$, $a<nb$, and $a=nb$.

3. Two couriers, A and B, are traveling the same road in the same direction, the former at rate a, the latter at rate b. They are at two places c miles apart at the same time. Where and when are they together?

SOLUTION AND DISCUSSION.—Let **XY** represent the road which the couriers are traveling in the direction from **X** to **Y**, and **A** and **B** the stations which they pass at the same time, *A* being at **A** when *B* is at **B**,, and **D** or **D**′ the place at which they are together. Call the distance from **B** to the place at which they are together $\pm x$, $+x$ when **D** is beyond **B**, and $-x$ when it is on the hither side of **A** and **B**, as at



D'. Then the distance from **A** to the point at which they are together is $c+(\pm x)$. Now disregarding the essential sign of x, and leaving it to be determined in the sequel, we have

$$\text{Distance } A \text{ travels from } \mathbf{A} = c+x,$$
$$\text{Distance } B \text{ travels from } \mathbf{B} = \quad x;$$

Time from passing **A** and **B** to the time they are together $\frac{c+x}{a}$ and $\frac{x}{b}$. But these are equal. Hence we are to discuss the equation.

$$\frac{c+x}{a}=\frac{x}{b}, \text{ or } x=\frac{bc}{a-b}, \text{ and } c+x=\frac{ac}{a-b}.$$

The points to be noticed in the discussion are, (1) when $a>b$, (2) when $a<b$, (3) when $a=b$, c being greater than 0 in each case but not ∞. Also the like cases when $c=0$.

When $c<0$ but not ∞.

We have, for $a>b$, x *positive*, which shows that the point at which they are together is at the right of **B**, *i. e.*, in the direction which they are traveling. The *time*, $\frac{x}{b}\left(\text{or } \frac{c+x}{a}\right)$, is *positive*, which shows that they are together *after* passing **A** and **B**.

For $a<b$, x is *negative*, and $c+x$, which equals $\frac{ac}{a-b}$, is also negative. This shows that they *were* together at a point at the left of **A**, that is, before they reached the stations **A** and **B**. With this the expressions for the time also agree. Thus $\frac{x}{b}$ becomes $-\frac{x}{b}$, and $\frac{c+x}{a}$ is also negative, since in this case $x>c$.

When $a = b$, $x = \frac{bc}{a-b} = \frac{bc}{0} = \infty$, and $c + x = \frac{ac}{a-b} = \frac{ac}{0} = \infty$ which indicates that they are never together.

When $c = 0$.

In this case $x = \frac{bc}{a-b} = 0$, and $c+x = \frac{ac}{a-b} = 0$, for a and b unequal, indicating that they are together when they are at **A** and **B**. This is evidently correct, since **A** and **B** coincide in this case. When $a = b$, $x = \frac{bc}{a-b} = \frac{0}{0}$, and $c+x = \frac{0}{0}$, which shows that they are *always* together, $\frac{0}{0}$ being a symbol of indetermination which in this instance may have any value whatever, as we see from the nature of the problem.

194. Sch.—The student should not understand that the symbol $\frac{0}{0}$ *always* indicates that the quantity which takes this form has an indefinite number of values. It is frequently so, but not necessarily. The indetermination may be only *apparent*, and what the value of the expression is must be determined from other considerations. The Calculus affords the most elegant general methods of evaluating such expressions. But the simple processes of Algebra will often suffice. Thus for $x = 1$, $\frac{1-x^3}{1-x} = \frac{0}{0}$. But $\frac{1-x^3}{1-x} = 1+x+x^2$, which, for $x = 1$, is 3. Hence $\frac{1-x^3}{1-x} = 3$, for $x = 1$. Here the apparent indetermination arises from the fact that the particular assumption (that $x = 1$) causes the two quantities between which we wish the ratio, viz., the numerator and denominator, to disappear. Let the student find that $\frac{1-x^5}{1-x+x^2-x^3} = 2\frac{1}{2}$ for $x = 1$. (See also ***190***, 3d part of demonstration.)

4. Two couriers starting at the same time from the two points A and B, c miles apart, travel *toward* each other at the rates a and b respectively. Discuss the problem with reference to the place and time of meeting. (Consider when $a>b$, $a<b$, and $a = b$.)

5. Two couriers, A and B, are traveling the same road in the same direction, the former at rate a, and the latter n times as fast. They are at two places c miles apart at the same time. Discuss the problem with reference to place and time of meeting as in Ex. 3, adding the considerations, $n > 1$, $n < 1$, $n = 1$, $n = 0$.

6. Divide 10 into two parts whose product shall be 40.

Solution and Discussion.—Let x and y be the parts, then $x + y = 10$, $xy = 40$, and $x = 5 \pm \sqrt{-15}$, $y = 5 \mp \sqrt{-15}$. These results we find to be imaginary. This signifies that the problem in its arithmetical signification is impossible: this indeed is evident on the face of it. But, although impossible in the arithmetical sense, the values thus found do satisfy the *formal*, or algebraic sense. Thus the *sum* of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ is 10, and the product 40.

7. The sum of two numbers is required to be a, and the product b: what is the maximum value of b which will render the problem possible in the arithmetical sense? What are the parts for this value of b?

8. Divide a into two parts, such that the sum of their squares shall be a minimum.

Sug's.—Let x and $a - x$ be the parts, and m the minimum sum. Then

$$x^2 + (a - x)^2 = 2x^2 - 2ax + a^2 = m;$$

whence $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{2m - a^2}$. From this we see that if $2m > a^2$ x is imaginary. Hence the least value which we can have is $2m = a^2$, or $m = \frac{1}{2}a^2$.

9. Divide a into two parts, such that the sum of the square roots shall be a maximum.

10. Let d be the difference between two numbers: required that the square of the greater divided by the less shall be a minimum.

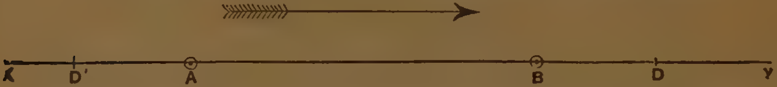
11. Let a and b be two numbers of which a is the greater, to find a number such that if a be added to this number, and b be subtracted from it, the product of this sum and this difference, divided by the square of the number, shall be a maximum.

SUG'S.—Let n be the number, and m the required maximum quotient. Then by the conditions $\frac{n^2 + (a-b)n - ab}{n^2} = m$, whence we find

$$n = -\frac{a-b}{2(1-m)} \pm \frac{\sqrt{a^2 + 2ab + b^2 - 4abm}}{2(1-m)}.$$

From this we see that the greatest value which m can have and render n real is $m = \frac{(a+b)^2}{4ab}$. This gives $n = -\frac{a-b}{2(1-m)} = \frac{2ab}{a-b}$.

12. To find the point on a line passing through two lights at which the illumination will be the same from each light.



SOLUTION.—Let A and B be the two lights, and XY the line passing through them. Let a be the intensity of the light A at a unit's distance from it, b the intensity of B at a unit's distance from it, c the distance between the two lights, as AB, and x the distance of the point of equal illumination from the light A, as AD (or AD'). Then, as we learn from Physics that the illuminating effect of a light varies inversely as the square of the distance from it, we have for the illumination of the point D by light A $\frac{a}{x^2}$, and for the illumination of the same point by light B, $\frac{b}{(c-x)^2}$. But by the conditions of the problem these effects are equal; hence we have the equation to be discussed; viz..

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}.$$

This gives $$\frac{(c-x)^2}{x^2} = \frac{b}{a};\ \text{or}\ \frac{c-x}{x} = \pm\sqrt{\frac{b}{a}} = \frac{\pm\sqrt{b}}{\sqrt{a}};$$

or $$\frac{c}{x} - 1 = \frac{\pm\sqrt{b}}{\sqrt{a}};\ \text{or}\ \frac{c}{x} = \frac{\sqrt{a}\pm\sqrt{b}}{\sqrt{a}};$$

or, finally, $$x = c\,\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}},\ \text{and}\ x = c\,\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}},$$

which are the values of x to be discussed.

Discussion.—I. *Let c be finite and* > 0.

1. When $a > b$, $x = c\,\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2}c$, since $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2}$ for $a > b$. This is as it should be, since for $a > b$ the point of equal illumination will evidently be nearer to B than to A. Again, the other value of x gives $x = c\,\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > c$, since $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is + and > 1, when $a > b$. Hence we learn that there is a point beyond B, as at D′, where the illumination is the same from each light.

If we assume $\sqrt{a} = 2\sqrt{b}$, AD $= \frac{2}{3}c$, and AD′ $= 2c$.

2. It is evidently unnecessary to consider the case when $a<b$, since this would only situate the points of equal illumination with reference to **A** as the preceding discussion does with reference to **B**.

3. When $a = b$,

$$x = c\,\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \tfrac{1}{2}c,$$

since, $$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}}{2\sqrt{a}} = \tfrac{1}{2}.$$

This is as it should be, since it is evident that in this case the point of equal illumination is midway between the lights. Again, for the second value of x, we have

$$x = c\,\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \infty.$$

This is also evidently correct; for when the lights are of equal intensity there can be no point beyond B, for example, at which the illumination from **A** will be equal to that from **B**, except when $x = \infty$, for which the illumination is 0 for each light. [Let the student give the reason.]

II. *When* $c = 0$. In this case the original equation $\frac{a}{x^2} = \frac{b}{(c-x)^2}$ becomes $\frac{a}{x^2} = \frac{b}{x^2}$, whence $a = b$. We then have

$$x = c\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} = 0;$$

and
$$x = c\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \frac{c\sqrt{a}}{\sqrt{\ }-\sqrt{b}} = \frac{0}{0}.$$

The former shows that there is a point of equal illumination where the lights are (when $c = 0$ they are together), and the latter shows that any point in the line is equally illuminated by each light. Both these conclusions are evidently correct.

SECTION VI.

PERMUTATIONS.

195. *Combinations* are the different groups which can be made of m things taken n in a group, n being less than m.

ILL.—Taking the 5 letters a, b, c, d, e, we have the 10 following *combinations* when the letters are taken 3 in a group, or, as it is usually expressed, taken 3 and 3: *abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.* Taken 2 and 2, we have the following 10 combinations: *ab, ac, ad, ae, bc, bd, be, cd, ce, de. It is to be noticed that no two combinations contain the same letters; i. e.*, they are *different* groups.

196. *Permutations* are the *different orders* in which things can succeed each other.

ILL.—Thus the two letters a, b, have the two permutations *ab, ba.* The three letters a, b, c have the 6 permutations *abc, acb, cab, bac, bca, cba.*

197. *Arrangements* are permutations of combinations.

ILL.—Taking the 10 combinations of 5 letters taken 3 and 3, and permuting each combination, we get the *arrangements* of 5 letters taken 3 and 3. Thus the combination *abc* gives the 6 arrangements *abc, acb, cab, bac, bca, cba.* In like manner each of the 10 combinations of 5 letters taken 3 and 3 will give 6 arrangements; whence, in all, 5 letters taken 3 and 3 have 60 arrangements.

198. *Prop.*—*The number of Arrangements of* m *things taken* n *and* n *is*

$$m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1).$$

DEM.—Let us consider the number of arrangements which can be made of the m letters a, b, c, d, etc., taken 2 and 2. Letting a stand first, we can have *ab, ac, ad*, etc., to $m-1$ arrangements. Letting b stand first, we can have *ba, bc, bd*, etc., to $m-1$ arrangements in each case, or $m(m-1)$ arrangements in all.

Again, *each* of these $m(m-1)$ 2 and 2 arrangements will give $m-2$ arrangements 3 and 3, by placing before it each of the letters not involved in it. Thus we have $m(m-1)(m-2)$ arrangements of m letters taken 3 and 3.

Once more, *each* of these $m(m-1)(m-2)$ 3 and 3 arrangements will give $m-3$ arrangements 4 and 4, by placing before it each of the letters not involved in it. Thus we have $m(m-1)(m-2)(m-3)$ arrangements of m letters taken 4 and 4.

Finally, we observe the law; *i. e.*, the number of arrangements is equal to the continued product of $m(m-1)(m-2)(m-3)\ldots\ldots\{m-(n-1)\}$ or $m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)$.

199. COR. 1.—*The number of Permutations of* m *things is*

$$1\cdot2\cdot3\cdot4\ldots\ldots m.$$

This is evident since arrangements become permutations when the number in a group is equal to the whole number considered; *i. e.*, when $n=m$.

200. Cor. 2.—*If* p *of the* m *letters are alike* (*as each* a), q *others alike,* r *others alike, etc., the number of permutations is*

$$\frac{1\cdot 2\cdot 3\cdot 4\,\ldots\ldots\, m}{\underline{|p}\times\underline{|q}\times\underline{|r}\times\text{ etc.}}.$$

Thus consider the permutations of *a, b, c, d,* viz., *abcd, bacd, acdb, bcda, acbd, bcad, abdc, badc, adcb, bdca,* etc. Suppose *b* to become *a*, then since for any particular position of *c* and *d*, as in *abcd*, there are as many permutations of the four letters as there can be permutations of the two letters *a* and *b*, viz., 1×2; if *b* becomes *a* there will be 1×2 fewer permutations when these two letters are alike than when they are different, *i. e.*, $\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2}$.

So, in general, if *p* of the letters are alike, there will be $1\cdot 2\cdot 3\ldots p$, or $\underline{|p}$ fewer permutations than if they are all different, etc.

201. Cor. 3.—*The number of Combinations of* m *things taken* n *and* n *is*

$$\frac{m\,(m-1)\,(m-2)\,(m-3)\,\ldots\ldots\,(m-n+1)}{1\cdot 2\cdot 3\cdot 4\,\ldots\ldots\, n}.$$

Since arrangements are permutations of combinations, the number of arrangements of *m* things taken *n* and *n* is equal to the number of combinations of *m* things taken *n* and *n* multiplied by the number of permutations of *n* things. Hence the number of combinations is equal to the number of arrangements of *m* things taken *n* and *n* divided by the number of permutations of *n* things.

EXAMPLES.

1. How many permutations can be made of the letters in the word *marble?* Of those in *home?* Of those in *logarithms?*

2. How many arrangements can be made of 10 colors taken 3 and 3? Of 7 colors taken 2 and 2? Taken 3 and 3? 4 and 4? 5 and 5? 6 and 6? 7 and 7? How many *mixtures* in each case, irrespective of proportions?

3. How many different products can be made from the 9 digits taken 2 and 2? 3 and 3? 4 and 4? 5 and 5? 6 and 6? 7 and 7? 8 and 8? 9 and 9?

4. How many different numbers can be represented by the 9 digits taken 2 and 2? 3 and 3? 4 and 4? etc.

5. In a certain district 3 representatives are to be elected, and there are 6 candidates. In how many different ways may a ticket be made up?

6. There are 7 chemical elements which will unite with each other. How many ternary compounds can be made from them? How many binary?

7. How many different sums of money can be paid with 1 cent, 1 3-cent piece, 1 5-cent piece, 1 dime, 1 15-cent piece, 1 25-cent piece, and 1 50-cent piece?

SUG.—If taken 1 and 1, how many? If 2 and 2, how many? If 3 and 3, etc.? How many in all?

8. In how many ways can 12 ladies and 12 gentlemen arrange themselves in couples?

9. If you are to select 7 articles out of 12, how many different choices have you?

10. How many different sums can be made from 1, 2, 3, 4, 5, 6, taken 2 and 2?

11. How many permutations can be made from the letters in the word *possessions?* (See **200.**) How many from the letters in the word *consistencies?*

12. How many different signals can be made with 10 different-colored flags, by displaying them 1 at a time, 2 at a time, 3 at a time, etc., the relative position of the flags with reference to each other not being taken into account?

[illegible]

[illegible] = 4 [illegible]

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